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Mechanics. — *The effective width of cylinders, periodically stiffened by circular rings.* By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of October 27, 1945.)

1. *Introduction.* The purpose of these few lines is to provide the reader with a simple rule for the computation of the greatest tangential stress, occurring in a thin-walled cylinder which in itself does only possess a slight flexural rigidity against alteration of its circular cross-section, and which therefore in constant distances ($= 2l$) has been stiffened by circular rings. The loadsystem of the cylinder is subject to the condition of periodicity in the axial direction, the period being identical with that of the rings; moreover the load is supposed to be symmetrical with respect to the plane of symmetry of two consecutive rings and to consist of radial and tangential components only (though axial components, if required, could as well be brought into account). A specialisation of a loadsystem of this kind presents itself, if all loads are concentrated in the planes of the rings and as a matter of fact in this paper attention will only be drawn to such a special system. The justification of this restriction must mainly be sought in its agreement with actual practice, but likewise in the fact that the stressproblem raised by a general loadsystem can be decomposed into two other problems, one of which refers to a cylinder of length $2l$, loaded in the prescribed way, but clamped at both ends, whilst the other one relates to the actual cylinder, exclusively loaded in the planes of its stiffening rings.

The orthodox method of solving this latter stress-problem consists in the separate treatment of the rings and the cylinder, the first ones under the action of unknown radial and tangential loadcomponents, the last one under the action of their reaction-components and the prescribed load-system.

The unknown loadcomponents can be found by equalizing the radial and tangential displacements of the corresponding points of the cylindrical shell and the stiffening rings, so that in principle the stress-distribution in the first one can be computed [1]). The authors used this method in solving a certain mineshaft-problem and had thereby full opportunity to take note of the laborious cipherwork which it involves. It was this very problem, which led them later on to the following approximative method, which, as they hope, may save a good deal of time and strain in the computation of stiffened cylinders.

If such a cylinder, stripped of its stiffening rings, is subject to an

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[1]) Comp. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, 1939; Springer Berlin, Ch. VI, where all required data can be found.

arbitrary loadsystem L in each of the former ring planes, it shows — if thin — a high degree of flexibility, characterized by large radial displacements. There exists, however, as will be proved in section 2, an infinite class of "characteristic" loadsystems L_p ($p = 1, 2, \ldots$) with radial and tangential components $r_p = B'_p \cos p\varphi$, $t_p = B''_p \sin p\varphi$ (φ designing the azimuthal coordinate) which, in their planes of application produce no radial displacements u_p but solely tangential displacements v_p, so that in these planes the cylinder feigns to lack any flexibility[1]). The circumferential elasticity of the cylinder can be measured by the quotient $v_p : t_p$, which for all points of the circular circumference appears to be a constant. All statements hitherto made are strictly confined to the planes of the loadsystems r_p, t_p and it must be emphasized, that in all other crosssections the state of stress and strain is a different one.

For the purpose of later comparison we now examine a circular ring of the same radius $\underline{a}$ and the same thickness $\underline{h}$ as the cylinder, and of the width $l' = \mu . a$. This ring too can be loaded by "characteristic" loads $\bar{r}_p = \bar{B}'_p \cos p\varphi$, $\bar{t}_p = \bar{B}''_p \sin p\varphi$ (but now uniformly distributed along the width l', so that the crossections of this ring remain plane), such that no radial displacements $\bar{u}_p$ occur and only tangential displacements $\bar{v}_p$ are present. Again it can be stated, that this ring, (which neither does bend in its plane) possesses a tangential elasticity to be measured by the quotient $\bar{v}_p : \bar{t}_p$, which for all points of the circumference proves to be constant. The width $\bar{l} = \mu . a$ is said to represent the effective width $\bar{l}_p = \mu_p . a$ of the cylinder (and the ring is said to be equivalent to the cylinder with respect to its tangential elasticity), if $\bar{v}_p : \bar{t}_p = v_p : t_p$. Consequently the cylinder and its equivalent ring of effective width $\bar{l}_p = \mu_p . a$ will have the same tangential displacements if loaded by characteristic loads (r_p, t_p), resp. $(\bar{r}_p, \bar{t}_p)$ the t-components of which are equal. It will be shown in sections 2 and 3 that the equality $t = \bar{t}_p$ involves — with a high degree of approximation — the equality $r_p = \bar{r}_p$, and therefore it can be said that the cylinder and its equivalent ring of effective width $\bar{l}_p = \mu_p . a$ behave similarly (with respect to the displacements u_p $(= 0)$ and v_p occurring in their midplanes) under the action of *equal* characteristic loads (r_p, t_p).

Before going on it may be stated, that — the magnitude of $\bar{l}_p$ depending upon the index p of the characteristic load — it will be convenient to restrict ourselves at the present to the consideration of one single characteristic load, characterized by a *fixed* value of the index p.

We now proceed by comparing a section of the *stiffened* cylinder with

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[1]) The amplitudes B'_p and B''_p of these characteristic components have a well-defined ratio; their absolute value can be fixed by any suitable "normalisation-condition".

an equally stiffened ring of effective width $\bar{l}_p = \mu_p . a$ as illustrated in fig. 1 (a) and 1 (b), and loaded by a system $r = A'_p \cos p\varphi$, $t = A''_p \sin p\varphi$,

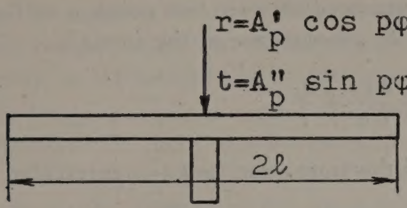

Fig. 1a

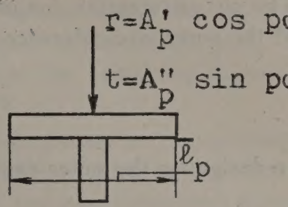

Fig. 1b

which differs from the characteristic loadsystem $r_p = B'_p \cos p\varphi$, $t_p = B''_p \sin p\varphi$, so that the ratio of the coefficients A'_p and A''_p differs from the ratio $B'_p : B''_p$.

On the assumption that the cylindrical shell of either of these constructions is devoid of all flexural rigidity, so that it cannot resist other loads than characteristic ones, it can be proved that both constructions are elastically identical, in so far as the transverse rings are similarly loaded at their joints with the cylindrical shell.

Firstly restricting ourselves to the stiffened cylinder and denoting by $r^* = C'_p \cos p\varphi$, $t^* = C''_p \sin p\varphi$ the load components to which the transverse ring is subject on its outer boundary, we find that the loads of the cylindrical shell are given by $(r - r^*)$, $(t - t^*)$; and our assumption leads to

$$r - r^* = a\, r_p \qquad t - t^* = a\, t_p \quad . \; . \; . \; . \; . \; . \; . \qquad (1)$$

or

$$A'_p - C'_p = a B'_p \qquad A''_p - C''_p = a B''_p \quad . \; . \; . \; . \; . \; . \qquad (1a)$$

where a denotes an unknown constant. If the corresponding quantities of the second construction (as far as they are distinct from those of the first one) are distinguished by a dash, we find in a similar manner

$$r - \bar{r}^* = \bar{a}\, r_p \qquad t - \bar{t}^* = \bar{a}\, t_p . \; . \; . \; . \; . \; . \; . \qquad (2)$$

or

$$A'_p - \bar{C}'_p = \bar{a} B'_p \qquad A''_p - \bar{C}''_p = \bar{a} B''_p \quad . \; . \; . \; . \; . \qquad (2a)$$

As to the transverse rings it can be proved [1], that the circumferential radial and tangential displacements u_p and v_p, caused by the loads $r^* = C'_p \cos p\varphi$ and $t^* = C''_p \sin p\varphi$ can be written as $u_p = u^*_p \cos p\varphi$, $v_p = v^*_p \sin p\varphi$ with

$$\left.\begin{array}{ll} u^*_p = \lambda_{11} C'_p + \lambda_{12} C''_p & \bar{u}^*_p = \lambda_{11} \bar{C}'_p + \lambda_{12} \bar{C}''_p \\ v^*_p = \lambda_{21} C'_p + \lambda_{22} C''_p & \bar{v}^*_p = \lambda_{21} \bar{C}'_p + \lambda_{22} \bar{C}''_p \end{array}\right\} \quad . \; . \; . \qquad (3)$$

[1] Comp. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, Springer Berlin 1939, Ch. VI, 6, p. 422.

where $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ are constants depending on the dimensions and the elastic properties of the rings.

The circumferential elongation measured between two points $\varphi = 0$ and φ of the outer circumference is — in consequence of the formula

$$\varepsilon_\varphi = \frac{1}{r}\frac{\partial v_p}{\partial \varphi} + \frac{u_p}{r},$$

— r designing the outer radius of the transverse ring — represented by:

$$\left.\begin{aligned}
&\int_0^\varphi \varepsilon_\varphi \,.\, r\,d\varphi = v_p + \int_0^\varphi u_p\,d\varphi = \\
&\qquad = \left[(\lambda_{11} C'_p + \lambda_{12} C''_p) - \frac{1}{p} (\lambda_{21} C'_p + \lambda_{22} C''_p) \right] \sin p\varphi \\
&\text{resp.:} \\
&\int_0^\varphi \bar{\varepsilon}_\varphi \,.\, r\,d\varphi = \bar{v}_p + \int_0^\varphi \bar{u}_p\,d\varphi = \\
&\qquad = \left[(\lambda_{11} \bar{C}'_p + \lambda_{12} \bar{C}''_p) - \frac{1}{p} (\lambda_{21} \bar{C}'_p + \lambda_{22} \bar{C}''_p) \right] \sin p\varphi
\end{aligned}\right\} \quad . \quad (4)$$

These elongations must — on account of the condition of consistency between the rings and their respective cylindrical shells — be equal to the corresponding circumferential elongation of the joining shells. The ratio of these latter elongations, however, is represented by $\alpha : \bar{\alpha}$ (the shells being "equivalent" and loaded by $(\alpha r_p, \alpha t_p)$ and $(\bar{\alpha} r_p, \bar{\alpha} t_p)$; comp. eq. (1) and (2)). Therefore it follows:

$$\begin{aligned}
&\bar{\alpha}\left[(\lambda_{11} - \frac{1}{p}\lambda_{21}) C'_p + (\lambda_{12} - \frac{1}{p}\lambda_{22}) C''_p \right] = \\
&\qquad = \alpha \left[(\lambda_{11} - \frac{1}{p}\lambda_{21}) \bar{C}'_p + (\lambda_{12} - \frac{1}{p}\lambda_{22}) \bar{C}''_p \right] \quad . \quad . \quad (5)
\end{aligned}$$

Substituting $C'_p, C''_p, \bar{C}'_p, \bar{C}''_p$ from (1*a*) and (2*a*) in this equation, we find:

$$(\bar{\alpha} - \alpha) \left[(\lambda_{11} - \frac{1}{p}\lambda_{21}) A'_p + (\lambda_{12} - \frac{1}{p}\lambda_{22}) A''_p \right] = 0 \quad . \quad . \quad . \quad (6)$$

from which it follows: $\bar{\alpha} = \alpha$ and consequently

$$\bar{r}^* = r^* \qquad \bar{t}^* = t^* \quad \text{q.e.d.}$$

Another method to prove this equivalence consists in the comparison of two constructions, as illustrated in figs. (1*a*) and (1*b*), for which the width $\bar{l}$ of the second one provisionally remains undetermined. Requiring that both constructions are similar with respect to their elastic behaviour of the

transverse rings, it follows immediately that C'_p and C''_p (resp. $\overline{C}'_p$ and $\overline{C}''_p$), and therefore r^* and $\overline{r}^*$ (resp. t^* and $\overline{t}^*$) must be equal. Then evidently $\overline{\alpha}$ must be equal to α, so that the condition imposed upon the cylindrical shells can be formulated by the identity of the tangential elasticities. This identity is effectuated, as has been proved in the foregoing exposition, by making the required width $\overline{l}$ equal to $\overline{l}_p$.

However the proof may be given, it is evident that the computation of the max. tangential stress occurring in the construction represented by fig. 1*a* can be replaced by the computation of the max. tangential stress in the T-shaped ring of fig. (1*b*) which can be performed by elementary means.

Returning to the stiffened cylinder, which in the planes of its stiffening girders is loaded by an *arbitrary* system L, we only have to expand L into a FOURIER-series, to compute — for every component $(A'_p \cos p\varphi + A''_p \sin p\varphi)$ — the maximal tangential stress which occurs in the T-shaped ring of corresponding effective width $\overline{l}_p = \mu_p . a$, and to sum up the results obtained in this way.

The aim of this treatise is to provide the reader with all necessary data with respect to the effective width $\overline{l}_p$, taking into account a rather wide range of practically occurring dimensions.

2. *The characteristic loads of a cylinder of length 2l.* We consider a cylinder of length $2l$, radius a, thickness h, loaded in its transverse plane of symmetry by

$$r_p = B'_p \cos p\varphi \qquad t_p = B''_p \sin p\varphi. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where, as before, r_p and t_p are defined per unit of circumferential length. If these loads are replaced by statically equivalent loads, defined by

$$\left.\begin{array}{lll} r_p = A'_p \cos p\varphi = \dfrac{B'}{2C} \cos p\varphi, & t_p = A''_p \sin p\varphi = \dfrac{B''}{2l} \sin p\varphi & \Big\} -c < z < c \\ r_p = 0 & t_p = 0 & z \begin{array}{l} < -c \\ > +c \end{array} \end{array}\right\} . \quad (2)$$

(z representing the distance to the transverse plane of symmetry) these latter ones can be expanded into FOURIER-series:

$$\left.\begin{array}{l} r_p = \sum\limits_{q=0}^{\infty} b'_{pq} \cos p\varphi \cos \lambda \dfrac{z}{a} \\ t_q = \sum\limits_{q=0}^{\infty} b''_{pq} \sin p\varphi \cos \lambda \dfrac{z}{a} \end{array} \quad \lambda = \frac{\pi q a}{l} \right\} \quad . \quad . \quad . \quad . \quad (2a)$$

The coefficients b'_{pq} and b''_{pq} are given by

$$b'_{po} = \frac{B'_p}{2l};\ b'_{pq} = \frac{\overline{B}'_p}{l} \frac{l}{k\pi c} \sin \frac{k\pi c}{l};\ b''_{po} = \frac{B''_p}{2l}\ b''_{pq} = \frac{\overline{B}''_p}{l} \frac{l}{k\pi c} \sin \frac{k\pi c}{l}. \quad (3)$$

The displacements belonging to the couple of loads $b'_{pq} \cos p\varphi \cos \lambda \frac{z}{a}$ and $b''_{pq} \sin p\varphi \cos \lambda \frac{z}{a}$ can be written as

$$\left.\begin{aligned} u_{pq} &= \overset{b}{u}_{pq} \cos p\varphi \cos \lambda \frac{z}{a} \\ v_{pq} &= \overset{b}{v}_{pq} \sin p\varphi \cos \lambda \frac{z}{a} \end{aligned}\right\} \text{ with } \left.\begin{aligned} \overset{b}{u}_{pq} &= a_{11}^{pq} b'_{pq} + a_{12}^{pq} b''_{pq} \\ \overset{b}{v}_{pq} &= a_{21}^{pq} b'_{pq} + a_{22}^{pq} b''_{pq} \end{aligned}\right\} . \qquad (4)$$

The coefficients a_{ij}^{pq} are rather complicated functions of p, q and the dimensions and elastic constants of the cylinder, which may be suppressed here [1]).

The *total* displacements u and v in the plane $z = 0$ can now be calculated from (2), (3) and (4), and amount to:

$$\left.\begin{aligned} u &= \sum_{q=0}^{\infty} (a_{11}^{pq} b'_{pq} + a_{12}^{pq} b''_{pq}) \cos p\varphi = \left[\left\{\frac{a_{11}^{po}}{2} + \sum_{q=1}^{\infty} a_{11}^{pq} \frac{l}{k\pi c} \sin \frac{k\pi c}{l}\right\} B'_p + \right. \\ &\qquad \left. + \left\{\frac{a_{12}^{po}}{2} + \sum_{q=0}^{\infty} a_{12}^{pq} \frac{l}{k\pi c} \sin \frac{k\pi c}{l}\right\} B''_p\right] \frac{\cos p\varphi}{2} \\ v &= \sum_{q+0}^{\infty} (a_{21}^{pq} b'_{pq} + a_{22}^{pq} b''_{pq}) \sin p\varphi = \left[\left\{\frac{a_{21}^{po}}{2} + \sum_{q=1}^{\infty} a_{21}^{pq} \frac{l}{k\pi c} \sin \frac{k\pi c}{l}\right\} B'_p + \right. \\ &\qquad \left. + \left\{\frac{a_{22}^{po}}{2} + \sum_{q=1}^{\infty} a_{22}^{pq} \frac{l}{k\pi c} \sin \frac{k\pi c}{l}\right\} B''_p\right] \frac{\sin p\varphi}{l} \end{aligned}\right\} . \qquad (5)$$

Now passing to the limit $c = 0$, we find for the loadsystem under consideration:

$$\left.\begin{aligned} u &= \left[\left\{\frac{a_{11}^{po}}{2} + \sum_{q=1}^{\infty} a_{11}^{pq}\right\} B'_p + \left\{\frac{a_{11}^{po}}{2} + \sum_{q=1}^{\infty} a_{12}^{pq}\right\} B''_p\right] \frac{\cos p\varphi}{l} \\ v &= \left[\left\{\frac{a_{21}^{po}}{2} + \sum_{q=1}^{\infty} a_{21}^{pq}\right\} B'_p + \left\{\frac{a_{22}^{po}}{2} + \sum_{q=1}^{\infty} a_{22}^{pq}\right\} B''_p\right] \frac{\sin p\varphi}{l} \end{aligned}\right\} . \qquad (5a)$$

The requirement $u = 0$ thereupon leads to:

$$\frac{B'_p}{B''_p} = \frac{a_{12}^{po} + \sum_{q=1}^{\infty} a_{12}^{pq}}{\frac{a_{11}^{po}}{2} + \sum_{q=1}^{\infty} a_{11}^{pq}} \quad \ldots \ldots \ldots \ldots \quad (6)$$

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[1]) They are to be found at full length in the Proceedings of the "Nederlandsche Akademie van Wetenschappen, Vol. XLIV, No. 5, 1941, p. 510".

and

$$\nu = \frac{\begin{vmatrix} \frac{a_{11}^{po}}{2} + \sum\limits_{q=1}^{\infty} a_{11}^{pq} & \frac{a_{12}^{po}}{2} + \sum\limits_{q=1}^{\infty} a_{12}^{pq} \\ \frac{a_{12}^{po}}{2} + \sum\limits_{q=1}^{\infty} a_{21}^{pq} & \frac{a_{22}^{po}}{2} + \sum\limits_{q=1}^{\infty} a_{22}^{pq} \end{vmatrix}}{\frac{a_{11}^{po}}{2} + \sum\limits_{q-1}^{\infty} a_{11}^{pq}} B_p'' \frac{\sin p\varphi}{l} \quad . \quad . \quad (7)$$

If we put

$$\frac{a_{ij}^{pq}}{2} + \sum_{q=1}^{\infty} a_{ij}^{pq} = \beta_{ij} \quad \begin{matrix} i=1,2 \\ j=1,2 \end{matrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the latter equations are transformed into:

$$B_p' : B_p'' = -\beta_{12} : \beta_{11} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$v = \frac{\begin{vmatrix} \beta_{11} \, \beta_{12} \\ \beta_{21} \, \beta_{22} \end{vmatrix}}{\beta_{11}} B_p'' \frac{\sin p\varphi}{l} = \beta_p B_p'' \frac{\sin p\varphi}{l} \quad . \quad . \quad . \quad . \quad (10)$$

To simplify the calculation of the coefficient β_p we furthermore put:

$$\left.\begin{aligned} a_{11}^{pq} &= \bar{a}_{11}^{pq} - p a_{12}^{pq} \\ a_{22}^{pq} &= \bar{a}_{22}^{pq} - \frac{1}{p} a_{12}^{pq} \end{aligned}\right\} \text{ and } \left\{\begin{aligned} \frac{\bar{a}_{11}^{po}}{2} + \sum_{q=1}^{\infty} \bar{a}_{11}^{pq} &= \bar{\beta}_{11} \\ \frac{\bar{a}_{22}^{po}}{2} + \sum_{p=1}^{\infty} \bar{a}_{22}^{pq} &= \bar{\beta}_{22} \end{aligned}\right\} \quad . \quad . \quad (11)$$

whereby we find:

$$\beta_p = \bar{\beta}_{22} + \frac{\bar{\beta}_{11}}{p^2} + \frac{\bar{\beta}_{11}^2}{p^2 (p \beta_{12} - \bar{\beta}_{11})} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The main contribution to the righthand side of this equation is due to

$$\beta_p' \equiv \bar{\beta}_{22} + \frac{\bar{\beta}_{11}}{p^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

whereas

$$\beta_p'' \equiv \frac{\bar{\beta}_{11}^2}{p^2 (p \beta_{12} - \bar{\beta}_{11})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

is rather small and could eventually be neglected.

Firstly paying attention to β_p', we find:

$$\left.\begin{aligned} \beta_p' &= \bar{\beta}_{22} + \frac{\bar{\beta}_{11}}{p^2} = \frac{\bar{a}_{22}^{po}}{2} + \frac{\bar{a}_{11}^{po}}{2p^2} + \sum_{q=1}^{\infty} \left[\bar{a}_{22}^{pq} + \frac{\bar{a}_{11}^{pq}}{p^2} \right] = \\ &= \frac{a_{22}^{po}}{2} + \frac{a_{11}^{po}}{2p^2} + \frac{a_{12}^{po}}{p} + \sum_{q=1}^{\infty} \left[a_{22}^{pq} + \frac{a_{11}^{pq}}{p^2} + 2 \frac{a_{12}^{pq}}{p} \right] \end{aligned}\right\} \quad . \quad (15)$$

Using the expressions, quoted from the above-mentioned treatise:

$$\left.\begin{aligned}
a_{11}^{pq} &= \frac{(\lambda^2+p^2)^2+k\,[3\lambda^4+2\,(1-\nu)\,\lambda^2\,p^2+p^4]}{N_{pq}}\;\frac{a^2}{B}\\
a_{12}^{pq} &= \frac{-p\,[\{(2+\nu)\,\lambda^2+p^2\}+k\,(2\lambda^4+2\lambda^2\,p^2+p^2)]}{N_{pq}}\;\frac{a^2}{B}\\
a_{22}^{pq} &= \frac{2(1+\nu)\lambda^2+p^2+k\left[\frac{2\lambda^6}{1-\nu}+\frac{5-\nu}{1-\nu}\lambda^4p^2+\frac{2(2-\nu)}{1-\nu}\lambda^2p^4+p^2(p^2-1)^2-\right.}{N_{pq}}\\
&\quad-\frac{\left.\frac{4\nu}{1-\nu}\lambda^4-\frac{2\,(2-\nu-\nu^2)}{1-\nu}\lambda^2p^2+\frac{2}{1-\nu}\lambda^2+p^2\right]}{N_{pq}}\;\frac{a^2}{B}
\end{aligned}\right\}\quad(16)$$

in which

$$\left.\begin{aligned}N_{pq}=(1-\nu^2)\,\lambda^4+k\,[\lambda^8+(4p^2-2\nu)\,\lambda^6+(6p^4-6p^2+4-3\nu^2)\,\lambda^4+\\+\{4p^6-2\,(4-\nu)\,p^4+2\,(2-\nu)\,p^2\}\,\lambda^2+p^4\,(p^2-1)^2\end{aligned}\right\}.\quad(17)$$

$$B=\frac{Eh}{1-\nu^2}\left(\nu=\frac{1}{m}\right),\; k=\frac{h^2}{12a^2}$$

the multinome

$$a_{22}^{pq}+\frac{a_{11}^{pq}}{p^2}+2\frac{a_{12}^{pq}}{p}$$

can be readily brought into the form

$$\frac{T'_{pq}}{N'_{pq}}\;\frac{a^2}{B}\quad.\;.\;.\;.\;.\;.\;.\;.\;.\;.\quad(18)$$

in which, — with $\frac{\pi a}{l}=\alpha$ —, T'_{pq} and N'_{pq} stand for:

$$\left.\begin{aligned}
T'_{pq}&=\frac{\alpha^{-2}\,q^{-2}}{kp^2}+\left[\frac{2}{1-\nu}+\left\{\frac{5-\nu}{1-\nu}p^2-\frac{4}{1-\nu}+\frac{3}{p^2}\right\}\alpha^{-2}q^{-2}+\left\{\frac{2(2-\nu)}{1-\nu}p^4-\right.\right.\\
&\quad\left.\left.-\frac{2\,(4-3\nu+\nu^2)}{1-\nu}p^2+\frac{2(2-2\nu+\nu^2)}{1-\nu}\right\}\alpha^{-4}\,q^{-4}+p^2\,(p^2-1)^2\,\alpha^{-6}\,q^{-6}\right]\\
N'_{pq}&=\frac{(1-\nu^2)}{k}\,\alpha^{-2}q^{-2}+\left[\alpha^2q^2+(4p^2-2\nu)+(6p^4-6p^2+4-3\nu^2)\alpha^{-2}q^{-2}+\right.\\
&\quad\left.+\{4p^6-2(4-\nu)\,p^4+2(2-\nu)p^2\}\;\alpha^{-4}q^{-4}+p^4(p^2-1)^2\alpha^{-6}q^{-6}\right]
\end{aligned}\right\}.\quad(19)$$

The coefficient β'_p (comp. 13) can now be calculated from

$$\beta'_p=\left[\frac{1}{2}\frac{T'_{po}}{N'_{po}}+\sum_{q=1}^{\infty}\frac{T'_{pq}}{N'_{pq}}\right]\frac{a^2}{B}=\left[\frac{1}{2p^2}+\sum_{q=1}^{\infty}\frac{T'_{pq}}{N'_{pq}}\right]\frac{a^2}{B}\quad.\;.\quad(20)$$

The computation of the secondary terms β''_p (comp. 14) requires that $\bar{\beta}_{11}$ and β_{12} be known. These quantities are defined by (11), (8) and (16), and prove to be equal to:

$$\left.\begin{aligned}\bar{\beta}_{11} = \sum_{q=1}^{\infty} \frac{T''_{pq}}{N'_{pq}} \frac{a^2}{B}, \text{ with } T''_{pq} = \Bigg[& \frac{1-\nu p^2 \alpha^{-2} q^{-2}}{k} + (3-2p^2) + \\ & + [2(1-\nu) p^2 - 2p^4] \alpha^{-2} q^{-2} \Bigg] \alpha^{-2} q^{-2}\end{aligned}\right\} . \quad (21)$$

$$\left.\begin{aligned}&\beta_{12} = -\frac{1}{2p(p^2-1)^2} \frac{1+k}{k} + \sum_{q=1}^{\infty} \frac{T'''_{pq}}{N'_{pq}} \frac{a^2}{B}, \text{ with } T'''_{pq} = \\ &= -p \left\{ \frac{[(2+\nu)\alpha^{-2}q^{-2} + p^2\alpha^{-4}q^{-4}]}{k} + [2+2p^2\alpha^{-2}q^{-2} + p^2\alpha^{-4}q^{-4}] \right\} \alpha^{-2}q^{-2}\end{aligned}\right\} . \quad (22)$$

It can be seen at once that the predominant term of β_{12} is the first one, k representing a very small number, lying between 10^{-4} and 10^{-7} in usual practice. Furthermore it can be controlled that $\bar{\beta}_{11}$ is small, and therefore — as has been already stated — β''_p will be small.

The practical calculation of the factor $\beta_p = \beta'_p + \beta''_p$ (comp. 10, 12, 13 and 14) requires the summation of the series

$$\sum_{q=1}^{\infty} \frac{T'_{pq}}{N'_{pq}}, \quad \sum_{q=1}^{\infty} \frac{T''_{pq}}{N'_{pq}} \text{ and } \sum_{q=1}^{\infty} \frac{T'''_{pq}}{N'_{pq}}. \quad \dots\dots \quad (23)$$

The way, in which this summation has been effectuated, will be elucidated at the hand of the first series, for which the summation of the consecutive fractions has been carried out to such value Q of the parameter q, that the remaining fractions safely can be approximated by

$$\frac{T'_{pq}}{N'_{pq}} \backsim \frac{\dfrac{\alpha^{-2} q^{-2}}{kp^2} + \dfrac{2}{1-\nu}}{\alpha^2 q^2}$$

and

$$\sum_{q=Q+1}^{\infty} T'_{pq} : N'_{pq}$$

on its turn can be approximated by

$$\sum_{q=Q+1}^{\infty} \frac{T'_{pq}}{N'_{pq}} \backsim \int_{Q+\frac{1}{2}}^{\infty} \frac{\dfrac{\alpha^{-2} q^{-2}}{kp^2} + \dfrac{2}{1-\nu}}{\alpha^2 q^2} \, dq.$$

It will later be seen for which values of $\alpha = \pi a/l$, k and p the coefficient β_p has been computated.

Finally we have to consider the ratio $-\beta_{12} : \beta_{11}$ (comp. (9)) by which

the characteristic load r_p, t_p (comp. 1 and 9), except for a constant factor, is defined. To this end we write:

$$-\frac{\beta_{12}}{\beta_{11}} = -\frac{\beta_{12}}{\bar{\beta}_{11} - p\beta_{12}}$$

and replace in consequence of the fact that $\bar{\beta}_{11}$ is small with respect to β_{12}, this ratio by

$$-\frac{\beta_{12}}{\beta_{11}} = \frac{1}{p} \quad \ldots \ldots \ldots \quad (24)$$

To give an idea of the degree of approximation, we mention, that for $\alpha = 1$, $p = 2$ and $k = 10^{-4}$ resp. 10^{-6}, we find as true values $-\frac{\beta_{12}}{\beta_{11}} =$ 0,4926 resp. 0,4997 instead of 0,5.

3. *The effective width.* We now consider a circular ring of radius a, thickness h and breadth $\bar{l}_p$, loaded by radial and tangential loadcomponents r and t per unit of circumferential length. Denoting the normal force, the shearing force and the bending moment occurring in an arbitrary cross-section (φ) by N, S and M, it can easily be verified, that the equations of equilibrium of an element of this ring run as follows:

$$\left.\begin{aligned} N - S' &= ra \\ N' + S &= -ta \\ M' + aN' &= -ta^2 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (1)$$

from which it can readily be concluded, that the non-existence of the bending moment M requires:

$$r = -t' \quad \ldots \ldots \ldots \quad (2)$$

Evidently this condition is fulfilled by

$$r_p = \frac{B_p''}{p} \cos p\varphi, \quad t_p = B_p'' \sin p\varphi \quad \ldots \ldots \quad (3)$$

The corresponding value of the normal force N amounts to

$$N = r_p \,.\, a = \frac{aB''}{p} \cos p\varphi \quad \ldots \ldots \ldots \quad (4)$$

and the tangential displacement is therefore given by

$$v = \int_0^{\varphi} \frac{N}{Eh\bar{l}_p} \, a \, d\varphi = \frac{a^2 B_p''}{pEh\bar{l}p} \int_0^{\varphi} \cos p\varphi \, d\varphi = \frac{a^2 B_p''}{p^2 Eh\bar{l}p} \sin p\varphi \quad . \quad (5)$$

Comparison of this displacement with the tangential displacement v (2, 10) of the cylinder of length $2l$ under the action of the same loadsystem —

provided at least that the approximation (2, 23) is accepted — leads to the following equation for the effective width:

$$\frac{\beta_p}{l} = \frac{a^2}{p^2 E h \bar{l}_p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As both components β'_p and β''_p of which β_r consists are multiples of a^2/B (comp. f.i. (2, 20)), we put:

$$\beta_p = \beta^*_p \cdot \frac{a^2}{B} = \beta^*_p (1-\nu^2) \frac{a^2}{Eh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and find thereby as final result for the effective width:

$$l_p (\equiv \mu_p a) = \frac{l}{(1-\nu^2) p^2 \beta^*_p}; \ \mu_p = \frac{l}{(1-\nu^2) p^2 \beta^*_p a}. \quad . \quad . \quad . \quad (8)$$

4. *The infinite cylinder.* For completeness-sake it is desirable to include in our computations the limiting case $l = \infty$. This can be done by the preliminary consideration of a cylinder of very great length. Restricting ourselves to the calculation of the coefficient β'_p (comp. 2, 20), it is seen at once, that the sum

$$S = \frac{1}{2} \frac{T'_{po}}{N'_{po}} + \sum_{q=1}^{\infty} \frac{T'_{pq}}{N'_{pq}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

can be interpreted as the l/π a fold of the surface, defined by a "staircase" curve, the steps of which have the height $T'_{pq} : N'_{pq}$ ($q = 0, 1, 2 \ldots$) and the width $\frac{\pi a}{2l}$ (belonging to the first step corresponding with $q = 0$) and $\pi a/l$ (belonging to all other steps). If l tends to $l = \infty$, the "staircase"-curve passes into a continuous one, the ordinate of which is represented by $T'_{pq} : N'_{pq}$, in which q is now to be regarded as a continuous parameter, varying from 0 to ∞.

Consequently we find

$$\beta'_p = \frac{l}{\pi a} \frac{a^2}{B} \int_{q=0}^{\infty} \frac{T'_{pq}}{N'_{pq}} dq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and — if β'_p be identified with β_p by neglecting the contribution of β''_p —

$$\bar{l}_p = \mu_p a \quad \mu_p = \frac{\pi}{(1-\nu^2) p^2} : \int_{q=0}^{\infty} \frac{T'_{pq}}{N'_{pq}} dq \quad . \quad . \quad . \quad . \quad (3)$$

We do not enter here into the calculation of the corrective term β''_p —

though in reality we have brought it into account — but it goes without saying, that in a similar way the sums

$$\sum_{q=1}^{\infty} \frac{T''_{pq}}{N'_{pq}} \frac{a^2}{B} \quad \text{and} \quad \sum_{q=1}^{\infty} \frac{T'''_{pq}}{N'_{pq}} \frac{a^2}{B}$$

which play a role in its valuation (comp. 2, 14, 21, 22), must be replaced by integrals.

All integrals to be considered can be reduced to the type:

$$\int_0^{\infty} \frac{A\lambda^2 + B}{\lambda^4 + P\lambda^2 + Q} \, d\lambda = \frac{B : \sqrt{Q} + A}{2\sqrt{2\sqrt{Q} + P}} \pi \quad . \quad . \quad . \quad . \quad . \quad (4)$$

5. *The results.* To provide the reader with a summary of the results obtained, graphs have been plotted from which — for any set of data and for any number p up to $p = 50$ — the coefficient μ_p can be read with a reasonable degree of approximation. In behalf of a more accurate computation of the effective width, the tables I, II, III, IV have been composed. Both graphs and tables relate only to four values of the parameter $k = h^2/12a^2$, viz. $k = 10^{-4}$, 10^{-5}, 10^{-6}, 10^{-7}. Obviously any particular case occurring in practice relates to another value of k. In such a case it will be recommendable to calculate (eventually by suitable interpolation) the required effective width, corresponding to the four values of $k = 10^{-4}$, 10^{-5}, 10^{-6}, 10^{-7}; to represent these values as ordinates in a graph, the abscis of which indicates $lg\ k$; to join the so obtained four points by a smooth curve, and to read from this curve the ordinate corresponding to the value of k under consideration.

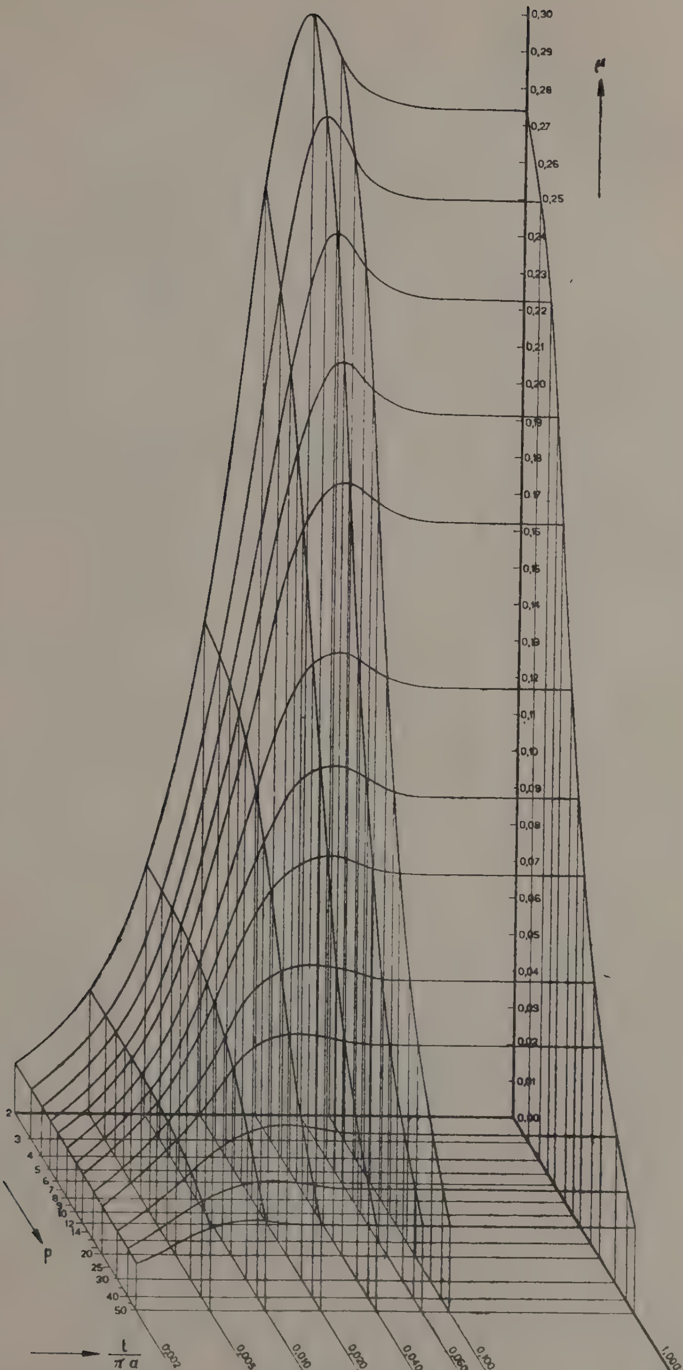
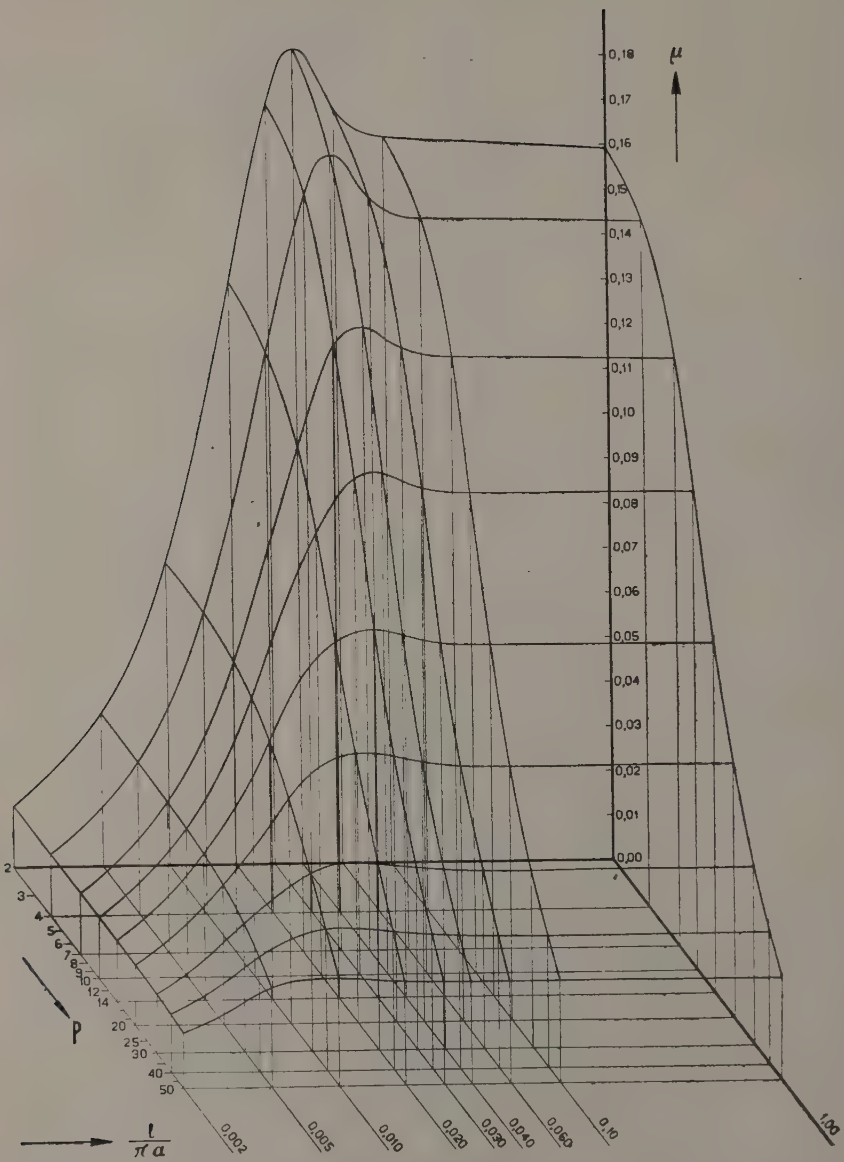

0,30
0,29
0,28
0,27
0,26
0,25
0,24
0,23
0,22
0,21
0,20
0,19
0,18
0,17
0,16
0,15
0,14
0,13
0,12
0,11
0,10
0,09
0,08
0,07
0,06
0,05
0,04
0,03
0,02
0,01
0,00
μ
2
3
4
5
6
7
8
9
10
12
14
20
25
30
40
50
p
t/πa
0,002
0,005
0,010
0,020
0,040
0,060
0,100
1,000

Fig. 2. $k = 10^{-4}$.



μ
0,18
0,17
0,16
0,15
0,14
0,13
0,12
0,11
0,10
0,09
0,08
0,07
0,06
0,05
0,04
0,03
0,02
0,01
0,00
2
3
4
5
6
7
8
9
10
12
14
20
25
30
40
50
P
$\frac{l}{\pi a}$
0,002
0,005
0,010
0,020
0,030
0,040
0,060
0,10
1,00

Fig. 3. $k = 10^{-5}$.

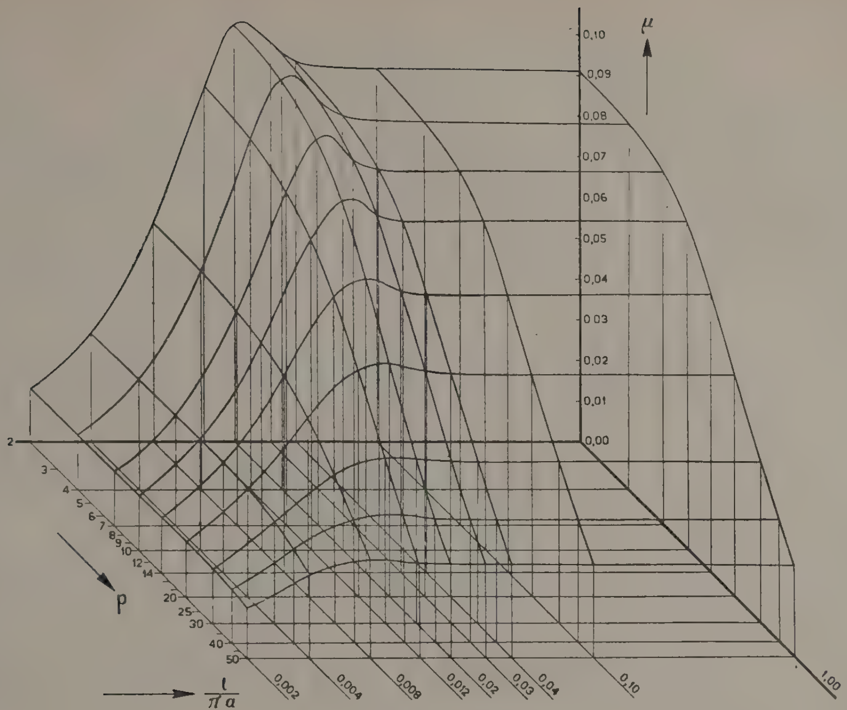


Fig. 4. $k = 10^{-6}$.

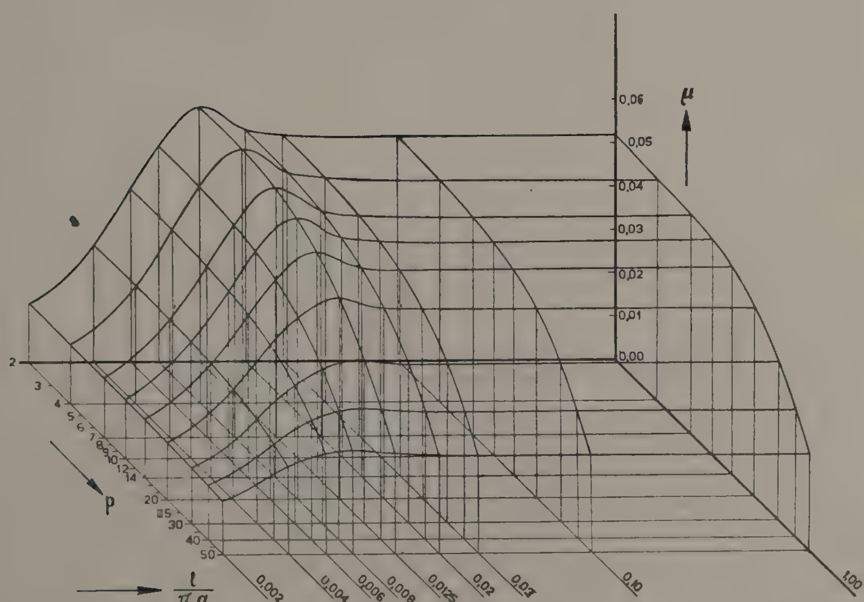


Fig. 5. $k = 10^{-7}$.

TABLE I. The effective width μ_p; $k = 10^{-4}$.

p \ $l/\pi a$	0,005	0,010	0,015	0,020	0,025	0,030	0,040	0,050	0,070	0,100	∞
2	0,0341	0,0680	0,1018	0,1342	0,1657	0,1975	0,2492	0,2818	0,3020	0,2886	0,2732
3	0,0341	0,0677	0,1008	0,1318	0,1612	0,1892	0,2317	0,2601	0,2783	0,2668	0,2567
4	0,0340	0,0673	0,0994	0,1287	0,1555	0,1802	0,2136	0,2362	0,2512	0,2422	0,2345
5	0,0340	0,0668	0,0979	0,1250	0,1489	0,1701	0,1967	0,2110	0,2206	0,2133	0,2074
6	0,0339	0,0661	0,0960	0,1206	0,1413	0,1593	0,1782	0,1870	0,1912	0,1850	0,1804
7	0,0338	0,0652	0,0940	0,1158	0,1336	0,1486	0,1613	0,1663	0,1678	0,1626	0,1589
8	0,0337	0,0644	0,0917	0,1108	0,1258	0,1378	0,1462	0,1491	0,1479	0,1435	0,1407
9	0,0335	0,0637	0,0890	0,1059	0,1183	0,1276	0,1331	0,1346	0,1319	0,1282	0,1260
10	0,0334	0,0626	0,0862	0,1009	0,1108	0,1173	0,1209	0,1219	0,1182	0,1150	0,1134
11	0,0333	0,0613	0,0827	0,0953	0,1032	0,1080	0,1100	0,1101	0,1069	0,1042	0,1029
12	0,0332	0,0602	0,0809	0,0918	0,0997	0,0997	0,1008	0,1003	0,0980	0,0959	0,0949
13	0,0331	0,0591	0,0770	0,0860	0,0907	0,0921	0,0922	0,0915	0,0897	0,0880	0,0872
14	0,0330	0,0580	0,0741	0,0814	0,0846	0,0860	0,0857	0,0846	0,0827	0,0815	0,0809
15	0,0329	0,0568	0,0716	0,0776	0,0798	0,0801	0,0797	0,0785	0,0770	0,0761	0,0755
16	0,0327	0,0554	0,0686	0,0734	0,0749	0,0750	0,0741	0,0729	0,0718	0,0717	0,0714
17	0,0324	0,0541	0,0660	0,0698	0,0709	0,0709	0,0695	0,0683	0,0677	0,0672	0,0670
18	0,0320	0,0531	0,0641	0,0667	0,0671	0,0670	0,0653	0,0641	0,0638	0,0635	0,0634
19	0,0317	0,0519	0,0612	0,0633	0,0636	0,0634	0,0617	0,0606	0,0604	0,0602	0,0601
20	0,0313	0,0506	0,0591	0,0604	0,0603	0,0600	0,0588	0,0578	0,0576	0,0575	0,0574
25	0,0303	0,0450	0,0493	0,0497	0,0493	0,0486	0,0484	0,0479	0,0476	0,0475	0,0474
30	0,0290	0,0400	0,0425	0,0420	0,0414	0,0410	0,0407	0,0403	0,0401	0,0401	0,0400
35	0,0277	0,0358	0,0363	0,0357	0,0351	0,0349	0,0347	0,0344	0,0343	0,0343	0,0342
40	0,0260	0,0317	0,0318	0,0313	0,0307	0,0304	0,0303	0,0301	0,0300	0,0300	0,0299
45	0,0241	0,0279	0,0273	0,0268	0,0265	0,0263	0,0262	0,0262	0,0262	0,0262	0,0261
50	0,0222	0,0242	0,0236	0,0232	0,0232	0,0232	0,0231	0,0231	0,0231	0,0230	0,0230

TABLE II. The effective width μ_p; $k = 10^{-5}$.

$l/\pi a$ / p	0,005	0,010	0,015	0,020	0,025	0,030	0,035	0,040	0,045	0,050	0,055	0,060	0,070	0,080	0,10	∞
	0,0341	0,0679	0,1010	0,1303	0,1550	0,1698	0,1803	0,1820	0,1805	0,1758	0,1718	0,1686	0,1650	0,1635	0,1623	0,1592
3	0,0341	0,0676	0,1000	0,1281	0,1511	0,1660	0,1739	0,1761	0,1741	0,1700	0,1662	0,1635	0,1607	0,1594	0,1587	0,1570
4	0,0340	0,0672	0,0988	0,1252	0,1465	0,1605	0,1673	0,1690	0,1673	0,1640	0,1609	0,1588	0,1563	0,1554	0,1550	0,1535
5	0,0340	0,0667	0,0964	0,1213	0,1406	0,1526	0,1580	0,1592	0,1577	0,1550	0,1527	0,1511	0,1492	0,1487	0,1483	0,1473
6	0,0339	0,0660	0,0946	0,1174	0,1346	0,1442	0,1486	0,1494	0,1480	0,1459	0,1441	0,1429	0,1414	0,1409	0,1405	0,1397
7	0,0338	0,0651	0,0921	0,1129	0,1279	0,1359	0,1394	0,1399	0,1387	0,1370	0,1357	0,1347	0,1335	0,1330	0,1327	0,1317
8	0,0337	0,0643	0,0896	0,1083	0,1212	0,1277	0,1304	0,1306	0,1295	0,1280	0,1270	0,1262	0,1253	0,1249	0,1246	0,1241
9	0,0335	0,0636	0,0870	0,1040	0,1141	0,1191	0,1212	0,1212	0,1202	0,1190	0,1182	0,1176	0,1168	0,1165	0,1162	0,1158
10	0,0334	0,0625	0,0847	0,0992	0,1072	0,1109	0,1124	0,1121	0,1112	0,1100	0,1094	0,1089	0,1084	0,1081	0,1079	0,1074
11	0,0333	0,0612	0,0822	0,0941	0,1000	0,1022	0,1032	0,1028	0,1020	0,1008	0,1003	0,0998	0,0994	0,0991	0,0988	0,0985
12	0,0332	0,0601	0,0797	0,0892	0,0933	0,0946	0,0952	0,0946	0,0939	0,0931	0,0927	0,0923	0,0920	0,0918	0,0916	0,0913
13	0,0331	0,0590	0,0770	0,0848	0,0878	0,0884	0,0885	0,0877	0,0871	0,0863	0,0860	0,0856	0,0854	0,0853	0,0852	0,0849
14	0,0330	0,0579	0,0742	0,0800	0,0822	0,0822	0,0820	0,0810	0,0804	0,0797	0,0794	0,0791	0,0790	0,0790	0,0790	0,0788
15	0,0329	0,0567	0,0718	0,0760	0,0777	0,0776	0,0771	0,0759	0,0754	0,0748	0,0745	0,0742	0,0741	0,0740	0,0740	0,0738
16	0,0327	0,0553	0,0687	0,0725	0,0736	0,0734	0,0727	0,0714	0,0710	0,0704	0,0701	0,0697	0,0696	0,0696	0,0696	0,0694
17	0,0324	0,0540	0,0665	0,0690	0,0700	0,0696	0,0687	0,0675	0,0671	0,0666	0,0664	0,0663	0,0662	0,0662	0,0662	0,0660
18	0,0320	0,0530	0,0639	0,0660	0,0666	0,0661	0,0652	0,0643	0,0639	0,0634	0,0632	0,0631	0,0630	0,0630	0,0630	0,0628
19	0,0317	0,0518	0,0612	0,0630	0,0633	0,0628	0,0621	0,0613	0,0610	0,0605	0,0603	0,0602	0,0601	0,0600	0,0600	0,0598
20	0,0313	0,0505	0,0590	0,0602	0,0601	0,0596	0,0591	0,0584	0,0581	0,0576	0,0574	0,0573	0,0572	0,0572	0,0572	0,0571
25	0,0303	0,0450	0,0493	0,0497	0,0493	0,0485	0,0484	0,0482	0,0480	0,0478	0,0477	0,0476	0,0475	0,0475	0,0475	0,0474
30	0,0290	0,0400	0,0425	0,0419	0,0413	0,0409	0,0407	0,0405	0,0403	0,0401	0,0400	0,0399	0,0399	0,0399	0,0399	0,0398
35	0,0277	0,0358	0,0363	0,0357	0,0351	0,0348	0,0347	0,0346	0,0345	0,0343	0,0342	0,0341	0,0340	0,0340	0,0340	0,0340
40	0,0260	0,0317	0,0318	0,0313	0,0307	0,0304	0,0303	0,0302	0,0301	0,0300	0,0300	0,0299	0,0299	0,0298	0,0298	0,0298
45	0,0241	0,0279	0,0273	0,0268	0,0265	0,0263	0,0262	0,0262	0,0262	0,0262	0,0262	0,0261	0,0261	0,0261	0,0261	0,0261
50	0,0222	0,0242	0,0236	0,0232	0,0232	0,0232	0,0231	0,0231	0,0231	0,0231	0,0231	0,0230	0,0230	0,0230	0,0230	0,0230

TABLE III. The effective width μ_p; $k = 10^{-6}$.

p \ $l/\pi a$	0,002	0,004	0,006	0,008	0,010	0,012	0,014	0,016	0,018	0,020	0,022	0,025	0,030	0,040	0,050	0,100	∞
2	0,0135	0,0270	0,0405	0,0538	0,0666	0,0781	0,0880	0,0955	0,1006	0,1039	0,1045	0,1035	0,0994	0,0937	0,0926	0,0918	0,0908
3	0,0135	0,0270	0,0405	0,0538	0,0663	0,0775	0,0870	0,0945	0,0996	0,1026	0,1033	0,1022	0,0980	0,0929	0,0919	0,0911	0,0904
4	0,0135	0,0270	0,0405	0,0538	0,0659	0,0769	0,0862	0,0934	0,0983	0,1012	0,1020	0,1007	0,0966	0,0920	0,0910	0,0904	0,0897
5	0,0135	0,0270	0,0405	0,0536	0,0655	0,0764	0,0853	0,0920	0,0966	0,0994	0,1003	0,0990	0,0951	0,0907	0,0898	0,0893	0,0888
6	0,0135	0,0270	0,0404	0,0531	0,0648	0,0750	0,0835	0,0898	0,0941	0,0968	0,0977	0,0965	0,0927	0,0888	0,0880	0,0876	0,0872
7	0,0135	0,0270	0,0402	0,0527	0,0640	0,0738	0,0819	0,0877	0,0917	0,0943	0,0951	0,0940	0,0904	0,0869	0,0862	0,0859	0,0856
8	0,0135	0,0270	0,0400	0,0522	0,0633	0,0726	0,0802	0,0855	0,0891	0,0915	0,0921	0,0912	0,0878	0,0850	0,0843	0,0840	0,0837
9	0,0135	0,0270	0,0397	0,0517	0,0624	0,0712	0,0783	0,0832	0,0865	0,0890	0,0895	0,0887	0,0856	0,0833	0,0828	0,0826	0,0823
10	0,0135	0,0270	0,0394	0,0511	0,0615	0,0698	0,0764	0,0808	0,0837	0,0857	0,0860	0,0854	0,0825	0,0809	0,0805	0,0803	0,0801
11	0,0135	0,0270	0,0390	0,0504	0,0604	0,0682	0,0742	0,0782	0,0808	0,0826	0,0828	0,0822	0,0795	0,0784	0,0781	0,0779	0,0777
12	0,0135	0,0270	0,0387	0,0499	0,0594	0,0667	0,0722	0,0758	0,0781	0,0797	0,0798	0,0793	0,0768	0,0758	0,0756	0,0755	0,0754
13	0,0135	0,0270	0,0383	0,0492	0,0583	0,0654	0,0704	0,0736	0,0756	0,0767	0,0767	0,0764	0,0741	0,0733	0,0732	0,0731	0,0730
14	0,0135	0,0270	0,0380	0,0486	0,0572	0,0636	0,0681	0,0709	0,0726	0,0736	0,0735	0,0731	0,0710	0,0706	0,0705	0,0704	0,0703
15	0,0135	0,0270	0,0376	0,0478	0,0559	0,0620	0,0660	0,0684	0,0698	0,0706	0,0705	0,0700	0,0681	0,0679	0,0679	0,0678	0,0677
16	0,0135	0,0270	0,0373	0,0469	0,0547	0,0602	0,0638	0,0659	0,0670	0,0675	0,0674	0,0669	0,0652	0,0651	0,0650	0,0649	0,0648
17	0,0135	0,0270	0,0369	0,0462	0,0535	0,0585	0,0617	0,0635	0,0644	0,0647	0,0645	0,0640	0,0624	0,0624	0,0624	0,0623	0,0622
18	0,0135	0,0270	0,0365	0,0454	0,0522	0,0568	0,0597	0,0612	0,0619	0,0619	0,0617	0,0612	0,0598	0,0598	0,0598	0,0597	0,0596
19	0,0135	0,0268	0,0363	0,0448	0,0512	0,0554	0,0580	0,0593	0,0598	0,0597	0,0595	0,0589	0,0577	0,0577	0,0577	0,0576	0,0575
20	0,0135	0,0266	0,0358	0,0440	0,0499	0,0538	0,0561	0,0572	0,0576	0,0573	0,0571	0,0565	0,0549	0,0549	0,0549	0,0548	0,0548
25	0,0135	0,0262	0,0338	0,0400	0,0441	0,0465	0,0477	0,0481	0,0481	0,0476	0,0474	0,0468	0,0463	0,0463	0,0463	0,0462	0,0462
30	0,0135	0,0251	0,0316	0,0366	0,0392	0,0406	0,0412	0,0413	0,0411	0,0406	0,0404	0,0399	0,0396	0,0396	0,0396	0,0395	0,0395
35	0,0133	0,0244	0,0299	0,0330	0,0346	0,0352	0,0354	0,0353	0,0350	0,0345	0,0344	0,0339	0,0339	0,0339	0,0339	0,0339	0,0339
40	0,0131	0,0231	0,0278	0,0301	0,0311	0,0313	0,0312	0,0310	0,0307	0,0304	0,0303	0,0299	0,0299	0,0299	0,0299	0,0299	0,0299
45	0,0128	0,0219	0,0256	0,0271	0,0275	0,0273	0,0270	0,0268	0,0266	0,0264	0,0263	0,0261	0,0261	0,0261	0,0261	0,0261	0,0261
50	0,0125	0,0204	0,0233	0,0243	0,0242	0,0238	0,0235	0,0233	0,0232	0,0232	0,0231	0,0230	0,0230	0,0230	0,0230	0,0230	0,0230

TABLE IV. The effective width μ_p; $k = 10^{-7}$.

p \ $l/\pi a$	0	0,002	0,004	0,006	0,008	0,010	0,0125	0,0150	0,02	0,03	0,05	0,1	∞
2	0	0,0136	0,0271	0,0400	0,0500	0,0566	0,0589	0,0575	0,0536	0,0524	0,0519	0,0515	0,0513
3	0	0,0136	0,0271	0,0400	0,0500	0,0565	0,0588	0,0574	0,0535	0,0523	0,0518	0,0514	0,0512
4	0	0,0136	0,0271	0,0400	0,0500	0,0563	0,0587	0,0572	0,0533	0,0522	0,0516	0,0513	0,0510
5	0	0,0136	0,0271	0,0398	0,0497	0,0560	0,0584	0,0569	0,0530	0,0518	0,0513	0,0511	0,0509
6	0	0,0136	0,0271	0,0397	0,0495	0,0556	0,0581	0,0566	0,0527	0,0515	0,0509	0,0507	0,0506
7	0	0,0136	0,0270	0,0395	0,0490	0,0552	0,0575	0,0560	0,0523	0,0511	0,0506	0,0505	0,0505
8	0	0,0136	0,0270	0,0392	0,0487	0,0547	0,0568	0,0553	0,0518	0,0507	0,0503	0,0503	0,0503
9	0	0,0136	0,0269	0,0389	0,0482	0,0542	0,0562	0,0547	0,0514	0,0503	0,0500	0,0500	0,0500
10	0	0,0136	0,0268	0,0386	0,0478	0,0534	0,0554	0,0540	0,0510	0,0499	0,0496	0,0496	0,0496
11	0	0,0136	0,0267	0,0384	0,0473	0,0526	0,0546	0,0532	0,0504	0,0494	0,0492	0,0492	0,0492
12	0	0,0136	0,0266	0,0382	0,0467	0,0518	0,0537	0,0524	0,0498	0,0489	0,0487	0,0487	0,0487
13	0	0,0136	0,0265	0,0379	0,0462	0,0511	0,0528	0,0516	0,0492	0,0483	0,0482	0,0482	0,0482
14	0	0,0136	0,0264	0,0376	0,0456	0,0503	0,0519	0,0507	0,0484	0,0477	0,0476	0,0476	0,0476
15	0	0,0136	0,0263	0,0373	0,0450	0,0495	0,0509	0,0498	0,0477	0,0471	0,0470	0,0470	0,0470
16	0	0,0135	0,0262	0,0370	0,0443	0,0487	0,0500	0,0489	0,0470	0,0464	0,0463	0,0463	0,0463
17	0	0,0135	0,0261	0,0367	0,0437	0,0479	0,0491	0,0480	0,0463	0,0457	0,0457	0,0457	0,0457
18	0	0,0135	0,0259	0,0363	0,0431	0,0471	0,0482	0,0472	0,0455	0,0450	0,0450	0,0450	0,0450
19	0	0,0135	0,0258	0,0360	0,0425	0,0462	0,0473	0,0463	0,0448	0,0444	0,0444	0,0444	0,0444
20	0	0,0135	0,0257	0,0356	0,0418	0,0453	0,0463	0,0455	0,0441	0,0437	0,0437	0,0437	0,0437
25	0	0,0134	0,0250	0,0338	0,0387	0,0414	0,0419	0,0412	0,0404	0,0403	0,0403	0,0403	0,0403
30	0	0,0132	0,0242	0,0318	0,0355	0,0373	0,0378	0,0370	0,0366	0,0366	0,0366	0,0366	0,0366
35	0	0,0130	0,0232	0,0299	0,0322	0,0337	0,0339	0,0333	0,0330	0,0330	0,0330	0,0330	0,0330
40	0	0,0128	0,0222	0,0278	0,0293	0,0303	0,0302	0,0297	0,0295	0,0295	0,0295	0,0295	0,0295
45	0	0,0126	0,0212	0,0256	0,0265	0,0271	0,0268	0,0264	0,0260	0,0260	0,0260	0,0260	0,0260
50	0	0,0125	0,0202	0,0232	0,0238	0,0238	0,0234	0,0230	0,0229	0,0228	0,0227	0,0227	0,0227

Petrology. — *The association of alkali rocks and metamorphic limestone in a block ejected by the volcano Merapi (Java).* By H. A. BROUWER.

(Communicated at the meeting of October 27, 1945.)

In 1928 I have drawn attention to the production of alkali rocks from pyroxeneandesitic magma associated with limestone [1]). Zones of trachyte, leucitephonolite and leucite-bearing phanerites were observed in a large block (50 × 50 × 30 centimetres) of metamorphic limestone, which was found on the lahar field of the Kali Batang on the SW slope of the volcano Merapi. I have recommended further study of limestone xenoliths from Javanese volcanoes but the limestone block from the Merapi still seems to be the only described example of its kind and it may be of interest to give some more details about the composition of the samples, which are at my disposal.

Numerous smaller xenoliths of metamorphic limestone have been found enclosed in the lava of the dome and flows of the Merapi. These smaller xenoliths are not associated with alkali rocks [1]). In the investigated samples of the large block the typical Merapi lava is not found; an original shell of this lava may have been lost when the block came rushing down the slope with great rapidity in one of the avalanches, which are a characteristic feature of the activity of the volcano.

The Merapi magma.

The rock of the lava dome, which started to rise in the crater in 1883 and which was partly destroyed during the explosive eruption of 1930 is a pyroxeneandesite with phenocrysts of plagioclase and pyroxenes: in some samples amphibole, which mostly is strongly resorbed, is also present. The numerous plagioclase phenocrysts (labradorite and bytownite) have a zonal structure with frequent alternation of basic and more acid zones. The phenocrysts of hypersthene are far less numerous and smaller than those of augite. Larger crystals of iron ore are also found. The groundmass consists of plagioclase, pyroxene, iron ore, some apatite and a varying amount of glass.

The rocks of the young lava flows of the volcano show a similar mineralogical composition and the chemical analyses of the dome and flows, which are given in the following table, show their uniform chemical composition.

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[1]) H. A. BROUWER. Alkaline rocks of the volcano Merapi (Java) and the origin of these rocks. Proc. Kon. Akad. v. Wet. Amsterdam, 31, 1928, p. 492 and Production of trachyte and phonolite from pyroxeneandesitic magma associated with limestone. Journ. of Geol., 36, 1928, p. 545.

	I	II	III	IV
SiO_2	54.81	54.81	55.55	54.95
Al_2O_3	19.26	18.90	18.92	18.98
Fe_2O_3	4.85	5.56	4.63	5.40
FeO	2.94	2.49	2.84	2.23
MnO	0.21	0.17	0.20	0.17
MgO	2.53	2.70	2.41	2.49
CaO	8.48	8.60	8.43	8.60
Na_2O	3.70	3.58	3.62	3.62
K_2O	1.95	2.15	2.13	2.13
H_2O+	0.28	0.13	0.15	0.21
H_2O-	0.05	0.01	0.06	0.11
TiO_2	0.78	0.84	0.80	0.83
P_2O_5	0.32	0.34	0.31	0.38
Sum	100.16	100.28	100.05	100.10

I. Pyroxeneandesite, eastern part of lava dome, Merapi. Anal. P. J. DEN HAAN in M. NEUMANN VAN PADANG. Vulk. en Seism. Med. Dienst v. d. Mijnbouw in Ned. Indië, no. 12, 1933, p. 76.
II. Pyroxeneandesite, eastern part of lava dome, Merapi. Same reference as. I.
III. Pyroxeneandesite, lava flow of 1930. Same reference as I.
IV. Pyroxeneandesite, lava flow of 1931. Same reference as I.

The normative composition was calculated [1]) for II with the following result:

Q	6.54	
Or	12.79	
Ab	30.39	
An	28.91	
Di	9.29	*Andose*
Hy	2.45	II, ''5, 3, 4
Mt	6.15	
Il	1.60	
Hm	1.36	
Ap	0.67	

The amount of normative quartz indicates that the glass is quaric. As alkali felspars were not observed amongst the constituents of the rock they may be concealed in the glass and in the plagioclase. In comparing

[1]) The normative compositions, which are mentioned in this publication, have been calculated by Dr. W. P. DE ROEVER.

the mode with the norm the modal composition of the soda-lime felspar leaves an amount of soda for alkali felspar.

In connection with the rather high potash content of the Merapi rocks it is of interest that rocks from other East Indian volcanoes, which have a similar chemical composition, are sometimes found associated with typical alkali rocks. Some chemical analyses are given for comparison in the following table.

	I	II	III	IV	V	VI
SiO_2	55.42	54.33	55.64	55.94	56.52	54.63
Al_2O_3	17.39	18.31	17.91	17·70	17.87	17.89
Fe_2O_3	1.56	3.01	3.19	2.76	3.69	5.73
FeO	6.82	4.41	5.13	4.74	1.91	3.40
MnO	0.71	0.17	0.14	0.13	0.17	0.15
MgO	3.28	3.65	3.20	3.50	4.18	3.19
CaO	7.57	8.41	7.16	7.60	6.61	7.82
Na_2O	2.41	3.04	3.65	3.66	3.72	3.25
K_2O	2.67	2.05	2.07	1.76	2.25	2.07
H_2O+	0.17	1.24	0.62	0.44	1.90	0.67
H_2O-		0.36	0.45	0.28	0.71	0.17
TiO_2	1.07	0.71	0.93	1.40	0.49	0.82
P_2O_5	0.58	0.13	0.15	0.28	0.22	0.17
Sum	99.98 [1])	99.82	100.24	100.19	100.24	100.04 [1])

I. Shoshonite, Mt Bromo, Java. Anal. E. W. MORLEY in J. P. IDDINGS and E. W. MORLEY. Journ. of Geol., 23, 1915, p. 233 (with 0.11 Cl, 0.03 F, 0.03 S, 0.13 BaO, 0.03 SrO).

II. Andesite (with augite, hyperstene, hornblende and olivine), Mt. Loeroes, Java. Anal. R. DJOKOJOEWONO in R. W. VAN BEMMELEN, Nat. Tijdschr. Ned. Indië, XCVIII, 1938, p. 194.

III. Pyroxeneandesite, Mt. Pangonan, Dieng Mts, Java. Anal. R. DJOKOJOEWONO in R. W. VAN BEMMELEN, Ingenieur Ned. Indië, 4, IV, 1937, p. 134.

IV. Hypersteneandesite, Mt Slamat, Java. Anal. R. G. REIBER in Jaarb. Mijnw. Ned. Indië, 59, Alg. Ged. 1930, p. 2.

V. Basalt (acid, with some hornblende, without olivine), Mt Penanggoengan, Java. Anal. C. KOOMANS in PH. H. KUENEN, Leidsche Geol. Med. 7, 1935, p. 283.

VI. Andesite to trachyandesite, Mt Soromandi, Soembawa. Anal. H. W. V. WILLEMS in H. A. BROUWER. Versl. Ned. Akad. v. Wet. 52, 1943, p. 304 (with 0.08 SrO).

The andesite of Mt Loeroes, Java (analysis II) is associated with leucitebasanites; the andesite to trachyandesite of Mt Soromandi, Soembawa (analysis VI) is associated with different leucite-bearing lavas and the association of the pyroxeneandesite of the Merapi with alkali rocks is shown by the local occurrence of these rocks in limestone.

[1]) With the additions mentioned below.

With regard to the composition of the glass base in the Merapi rocks the analyses [1]) of a glass-rich pyroxeneandesite and its glass base (I and II in the following table) are of particular interest. The rock was collected in 1876 and may have been thrown out during the strongly explosive eruption of 1872.

	I	II	III
SiO_2	57.76	65.05	53.95
Al_2O_3	18.39	17.60	27.45
Fe_2O_3	7.51	3.10	1.09
CaO	6.21	3.58	11.48
MgO	3.34	1.05	trace
Na_2O	3.63	3.54	4.51
K_2O	2.61	4.16	0.79
H_2O [2])	0.94	1.56	0.48
Sum	100.89	99.64	99.75

I. Pyroxeneandesite (vitroandesite), Mt Merapi, Java. Anal. A. LAGORIO, Tscherm. Min. u. Petr. Mitt., 8, 1887, p. 467.
II. Glass base in I. Same reference as I.
III. Plagioclase phenocrysts in I. Same reference as I.

The plagioclase phenocrysts (III) have about the composition $Ab_3\ An_4$ and contain about 4.5 % Or. A comparison of the analyses I and II shows that the residual liquid was richer in silica and potash and poorer in iron, lime and magnesia. The calculation of the normative composition of the glass base gives about 20 % Q, 25 % Or, 30 % Ab and 18 % An and shows that the glass is strongly oversaturated and conceals much felspar, especially potash-rich felspar, which was not found by microscopical investigation. Although enriched in potash, the residual liquid would need strong desilication before it could have crystallized to the leucite-bearing mineral assemblages, which are found in the ejected limestone block.

The metamorphic limestone.

The block of metamorphic limestone has been described as a greyish green "schist" interlarded with small lenticles and strings of limestone. The limestone of the lenticles is altered by contactmetamorphism and partly consists of a chalky lime powder [3]).

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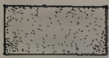
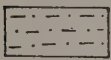
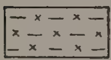
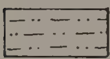
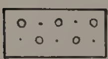
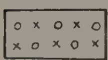
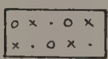
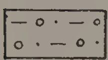
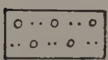
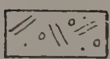
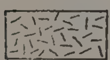
[1]) A. LAGORIO. Über die Natur der Glasbasis, sowie der Kristallisationsvorgänge im eruptiven Magma. Tsch. Min. u. Petr. Mitt., 8, 1887, p. 467.

[2]) Loss on ignition.

[3]) G. L. L. KEMMERLING. De hernieuwde werking van den vulkaan G. Merapi (Midden-Java) van begin Augustus 1920 tot en met einde Februari 1921. Vulk. Meded. Dienst van het Mijnw. in Ned. Oost-Indië, No. 3, 1921, p. 29.

REFERENCES.

Principal minerals of the assemblages in figures 1—7.

	limestone.
	wollastonite (locally gehlenite, pyrrhotite).
	wollastonite, augite.
	gehlenite, augite, garnet, wollastonite.
	plagioclase, augite (often with titanite).
	plagioclase, biotite.
	plagioclase, wollastonite, augite.
	potash-rich felspar, augite.
	potash-rich felspar, augite, biotite.
	potash-rich felspar, plagioclase, augite.
	potash-rich felspar, wollastonite, augite.
	leucite, augite.
	leucite, biotite.
	leucite, biotite, augite.
	leucite, plagioclase, augite.
	leucite, augite, wollastonite.
	leucite, potash-rich felspar, augite.
	glass-bearing parts (mainly leucitephonolite and trachyte to trachyandesite).
	calcite veins.

The figures 1—7, which are all six times enlarged, are sketches illustrating the composition of the investigated samples.

Figs. 1 and 2.

These figures show the transformation of the limestone into wollastonite and gehlenite and the formation of plagioclase- and leucite-bearing mineral assemblages.

The wollastonite-rich mineral assemblages are connected by transitions

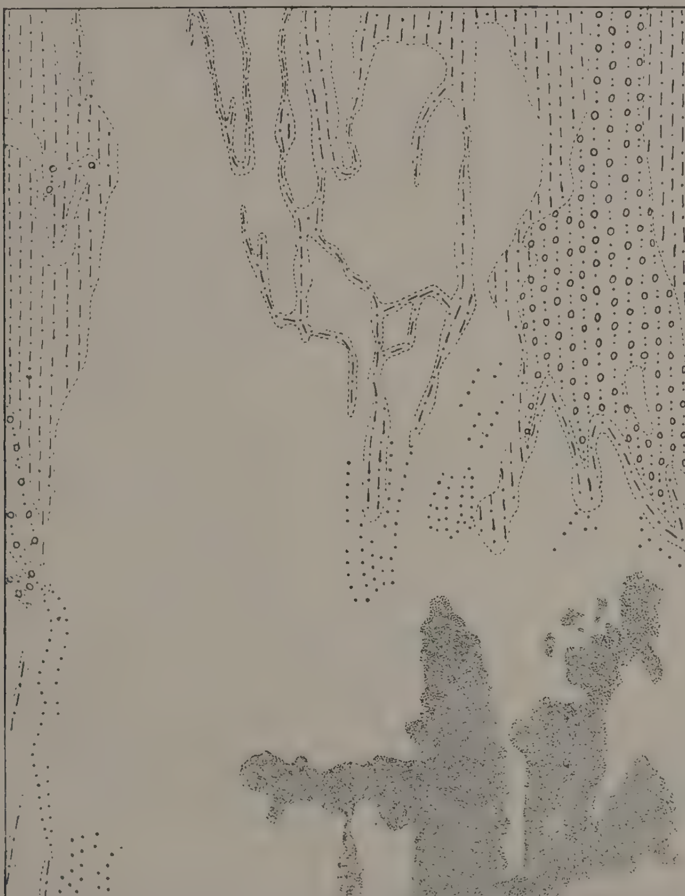


Fig. 1. Enl. × 6.

to those which mainly consist of plagioclase and augite and to those which mainly consist of leucite and augite. In the figures the following assemblages are distinguished: wollastonite-augite, plagioclase-wollastonite-augite, plagioclase-augite, leucite-wollastonite-augite and leucite-augite [1].

[1]) The colourless mineral, which generally shows distinct optical anomalies, is always

Microscopically the limestone shows slight recrystallization and where the transformation into lime silicates is starting its colour changes in narrow zones and spots to dark grey, brown or black (fig. 8).

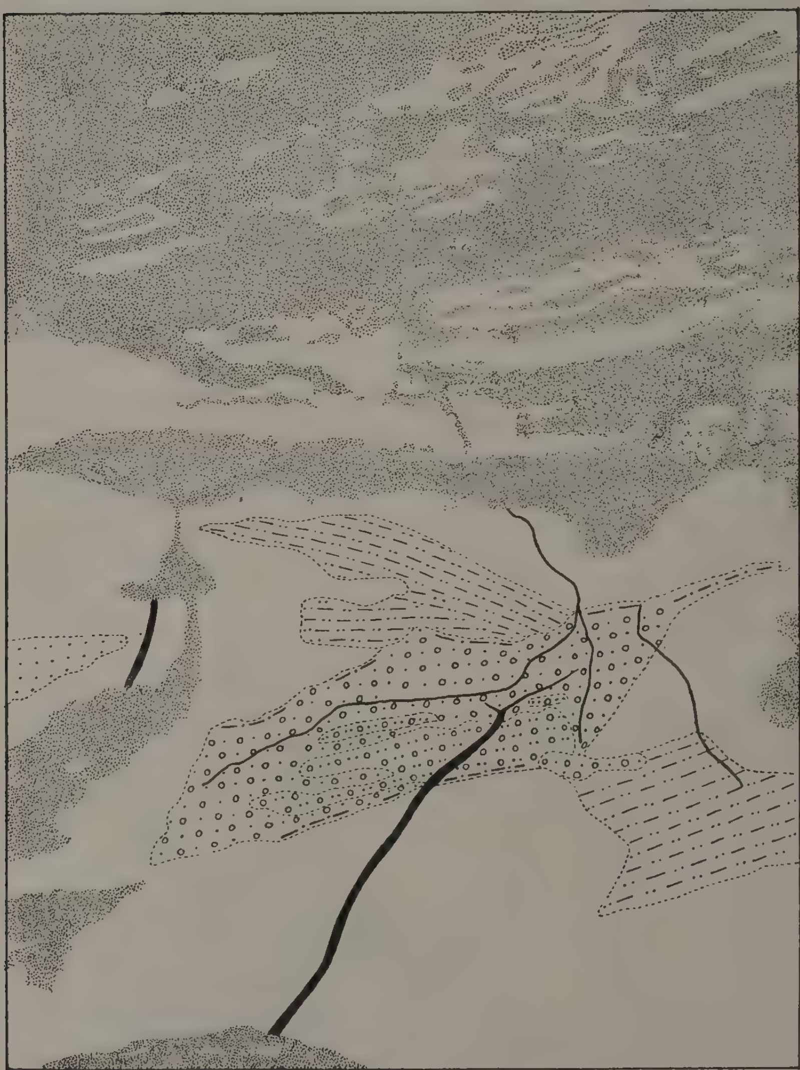


Fig. 2. Enl. × 6.

In the wollastonite-rich assemblages the crystals of this mineral may attain a length of 0.6 mm and the larger crystals are enveloped in a fine-

mentioned in this publication as leucite. Several microchemical tests and measurements of the refractive index confirm this determination; it is however possible that analcime (or potash analcime) is not absent in the investigated samples.

grained mixture of wollastonite and gehlenite with a grain size of about 0.03—0.06 mm. The small crystals of gehlenite accumulate locally in spots and bands, which sometimes border upon the plagioclase-bearing mineral assemblages. Other constituents of the wollastonite-rich assemblages are pyrrhotite, of which the size reaches up to 0.2 mm, and minute, mostly irregular, crystals, which resemble titanite and perowskite.

The other assemblages all contain augite, which appears in varied shades of green and brown, passing into greyish and purplish grey tints. The augite sometimes shows a purplish grey core with a narrow greenish to brownish grey margin. The crystals are often anhedral, their size may reach up to 0.4 mm, it varies gradually and very small grains (0.003—0.03 mm) are frequently found. The plagioclase in the plagioclase-augite and plagioclase-wollastonite-augite assemblages is up to about 2 mm in length (fig. 11), the crystals mostly are of rather large size except in the left upper part of fig. 1 where the grain size of the major part is about 0.01—0.03 mm and larger crystals, which reach up to 0.2 mm, are seldom found. The plagioclase is mostly very basic (bytownite to anorthite), a zonal structure with a rim of more acid plagioclase was observed in some crystals. For the greater part the crystals are anhedral, they enclose other constituents of the assemblages, especially augite, and they contain numerous minute inclusions of liquid or gas. The leucite forms a base, in which augite or augite and wollastonite are embedded; it shows distinct optical anomalies with polysynthetic twins. Other constituents of minor importance are pyrrhotite, titanite and calcite. The occurrence of calcite in narrow veins is shown in fig. 2.

Fig. 3.

A cross-section of a part of the metamorphic margin of a limestone lenticle is seen in the left lower corner of the figure. Wollastonite-rich assemblages are shown in the right half and in the left upper corner. The mineral assemblages in the main part of the figure vary strongly in mineralogical composition and grain size. All the glass-bearing parts, although showing important differences, are indicated by the same signature.

The limestone of the lenticle passes outward into a zone with a thickness of some millimetres, which for the greater part consists of lime sillicates: in the inner part mostly wollastonite and farther outward mostly gehlenite. A small part of this zone appears in the left lower part of the figure. It is surrounded by a zone in which the chief minerals are gehlenite, augite, wollastonite and garnet. Calcite, pyrrhotite, magnetite and small grains, resembling titanite and perowskite, are also found. In this zone the gehlenite crystals are partly of larger size, they reach up to 1.5 mm in diameter. The maximum size of the wollastonite is 0.2 mm. The mostly greenish augite may attain a length of nearly 1 mm. Some crystals of augite have a purplish core, which has the same crystallographic orientation and seems to be the

remnant of an originally larger crystal, which has been replaced by the greenish augite. Intergrowth textures, e.g. of light yellow garnet and

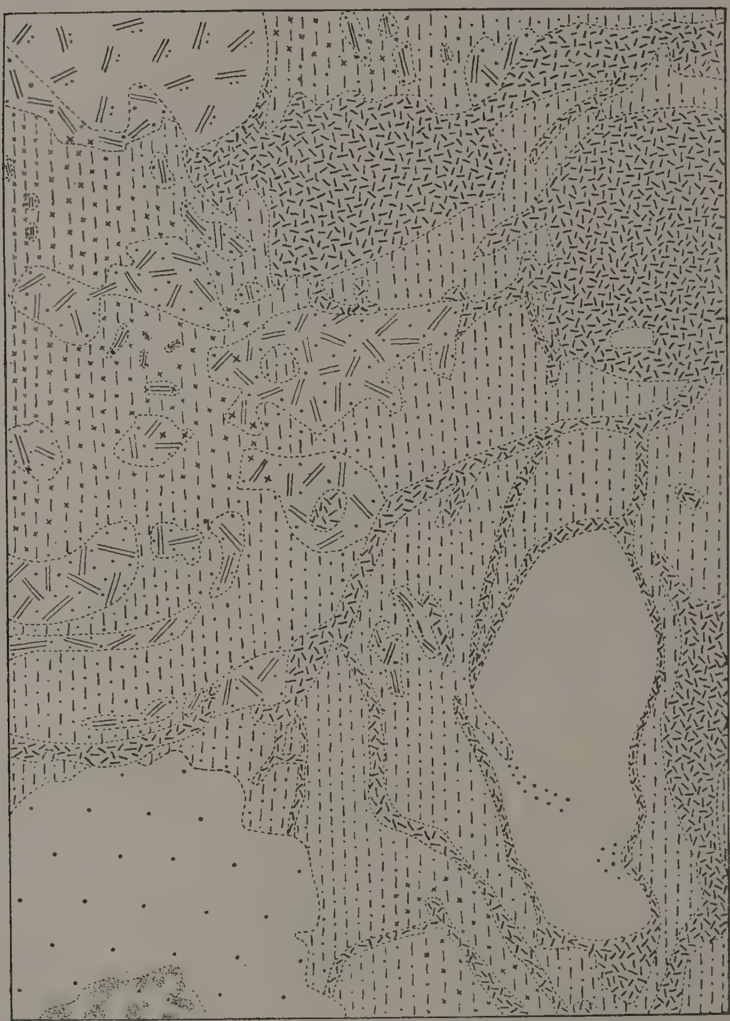


Fig. 3. Enl. × 6.

greenish to brownish grey augite, and poikiloblastic textures are a common feature. Characteristic is also the more or less distinct radiate arrangement in protruding sectors of the outer zone, which indicates a divergent streaming out of substance towards the contact. This outward movement is also illustrated by crystals of purplish augite, which partly lie outside the lime silicate zone and of which the part within this zone is partly transformed into greenish augite. At the contact the lime silicate zone is often rich in calcite

and sometimes calcite and greenish augite are the only constituents (fig. 9). Other thin sections of the marginal part of the same limestone lenticle show that the character of the lime silicate zones changes from point to point. One section for instance shows a 2—3 mm thick wollastonite-rich inner zone, which is followed outward by a 0.2—0.5 mm thick wollastonite-gehlenite zone and a 0.05—0.75 mm thick outer zone consisting of calcite and augite.

The wollastonite-rich assemblages in the right half of the figure consist almost entirely of this mineral; the crystals reach a length of 0.5 mm but generally they are much smaller (about 0.1 mm). Augite is found locally, it is light-coloured with a greenish grey tint and larger crystal skeletons reach up to 1.5 mm in size. Calcite is found especially in the marginal parts, where the wollastonite borders upon a glass-rich zone. In the wollastonite-rich assemblage in the left upper corner of the figure the skeletons of augite, which often consist of isolated parts, attain dimensions up to 2 mm. The small crystals of potash-rich felspar are up to 0.12 mm in diameter. Calcite and some rare grains resembling titanite are also found. The two last mentioned wollastonite-rich assemblages have been attacked and imbibed by the surrounding substances and there are no indications that the line of contact moved outward. An outward movement seems to be connected with the presence of a core of limestone in the lime silicate assemblages.

The other assemblages in the figure — as far as they do not belong to the glass-bearing varieties — are microcrystalline to fine grained mixtures in which different felspars, augite and biotite are the main constituents. They often are vesicular, the vesicles are empty or they contain calcite. In the larger and smaller spots, which are indicated schematically in the left part of the figure, the principal minerals are potash-rich felspar and light greenish to light brownish grey augite (fig. 13). The smallest spots are not shown in the figure. Biotite is found in some of the spots but only where the surrounding mineral assemblage also contains biotite. It is strongly pleochroic from very dark brown to light yellow. The grain size is up to 0.5 mm for potash-rich felspar, up to 0.6 mm for augite and up to 0.1 mm for biotite. Other constituents which are sometimes found are pyrrhotite, titanite and calcite. An interstitional glass-bearing substance is sometimes found locally. The spots lie in a mixture with a much smaller grain size in which basic plagioclase is the main constituent, while brown biotite is abundant in one part and absent in the other. The biotite and most of the plagioclase reach up to about 0.05 mm in size. Pyrrhotite, magnetite and minute grains of augite and iron ore are also found. A colourless residual cement is present in varying amount. If birefrigent the refractive index points to potash-rich felspar; the cement may partly consist of leucite. In another thin section made of the same rock fragment at a distance of two centimetres from the described section, anhedral leucite is found as an important constituent of cement and spots. The biotite-free part is characterized by the presence of some-

what larger crystals of titanite and grey to greenish or purplish grey augite, which may attain a diameter of 0.1 mm and 0.25 mm respectively. In some parts of the plagioclase-augite assemblage, which are not shown by a separate signature in the figure, the grain size of the plagioclase is larger, reaching up to 0.2 mm. In these parts potash-rich felspar is sometimes found. Two spots, in which both felspars have a grain size up to about 0.1 mm, are shown in the figure. As to the shape of the crystals it is clear that biotite and plagioclase have been able to develop the best crystal boundaries. Euhedral crystals of biotite are numerous and the shape of the plagioclase is often rectangular. They have hindered the growth of crystals of potash-rich felspar and augite and they are found enclosed in larger crystals of the last mentioned minerals. In the spots, where augite and potash-rich felspar have developed under circumstances of comparative freedom, the augite may partly exhibit good crystal boundaries.

The glass-bearing parts of the rock show a great variety as to their glass content and the size and shape of the mineral constituents. There are transitions between these parts and the microcrystalline mixtures and then the boundary lines in the figure are approximate. The glass-bearing assemblages are strongly vesicular, the vesicles are empty or they contain calcite. In the more or less rounded parts in the upper part of the figure the vesicles, which mostly are empty, take up much of the space (fig. 10). The remaining part contains about equal quantities of a light coloured greenish augite and a brownish to colourless glass, which is slightly devitrified and contains some lath-shaped microlites of potash-rich felspar with a length of about 0.06 mm. Large skeletons of augite reach up to 1.5 mm in size, they often consist of isolated parts with simultaneous extinction. Smaller, often euhedral, crystals of augite with a size of 0.05—0.1 mm are also found. Near the margin of these glass-rich parts a zone, which is developed locally, contains small lath-shaped crystals of basic plagioclase, which partly have acid rims. This zone forms a transition to the adjacent plagioclase-rich mineral assemblage. The narrow zone which surrounds the wollastonite rock in the right lower part and the small oval spot in the centre of the figure show a similar character.

The zone to the right of the wollastonite rock in the right lower part of the figure is also strongly vesicular. It is much poorer in augite of which the crystals reach up to 0.25 mm in size. The microlites in the brownish glass are more numerous, they mostly consist of potash-rich felspar in which a core of basic plagioclase with extinction angles up to 37° is often found. Some plagioclase is also found in separate microlites and their number increases in a marginal zone, which is developed locally.

The other glass-bearing parts all form narrow bifurcating zones or veins in which the amount of glass is much smaller. They contain numerous vesicles, which sometimes take up more than half of the space, and mostly are empty. Microlites of potash-rich felspar, often with a core of basic plagioclase, are numerous. The average size of the microlites is about

0.05 mm, the diameter of the cores of plagioclase is rarely more than 0.01 mm. Minute crystals of light greenish augite and iron ore are found in the interstitional glassy substance, while a brownish mineral, which was rarely observed, resembles the kataphoritic amphibole of the leucite-bearing zones, which are found in other parts of the rock (p. 182). Phenocrysts are mostly absent. Numerous lath-shaped phenocrysts of potash-rich felspar with a length up to 1 mm were observed in one place in the left lower part of the figure. They enclose small plagioclase and augite crystals. Titanite and larger augite crystals are also sometimes found. As the potash-rich felspar, this augite has developed under circumstances of comparitive freedom and has been able to develop better crystal boundaries if compared with those of the adjacent plaglioclase-augite assemblage. The transition between the glass-bearing zone and the adjacent mineral assemblages is particularly clear where crystals of augite or titanite are partly or entirely enclosed in the glass-bearing zone and where this zone contains separate lath-shaped crystals of plagioclase.

The general features shown by the different mineral assemblages are the remarkable variability of grain size and mineralogical composition and the abundance of vesicles in the assemblages found between the parts of the rock, which are rich in wollastonite. Wollastonite must have occupied much of the space where these assemblages are now found and it has been attacked and replaced by magmatic substances which were rich in volatile constituents.

Fig. 4.

This figure shows the arrangement in more or less parallel zones of different mineral assemblages. Limestone appears in the lower part and a partly altered crystal of garnet is seen in the central part of the figure. The wollastonite-rich assemblage contains pyrrhotite and gehlenite. The crystals of wollastonite reach a size up to 1 mm but smaller grains with an average size of about 0.1 mm are numerous. Pyrrhotite mostly forms irregular crystals, which are up to 0.1 mm in diameter. Gehlenite is found in crystals, which reach up to 0.25 mm in size; it is found locally and is for instance lacking in the right middle part of the figure. A gehlenite-rich zone is found along a part of the lower boundary of the plagioclase-augite-leucite assemblage in the left lower part of the figure.

The garnet of which one crystal occurs in the central part of the figure is a nearly colourless light pink grossularite. One larger and two smaller separate parts of the original crystal are nearly entirely surrounded by a rim in which small crystals of a light brown garnet are found in wollastonite. The size of the small garnet crystals varies between 0.005 and 0.1 mm, it mostly is about 0.03 mm. These crystals, which are roughly shown in the figure, are also found scattered outside the rim but are never found in the adjacent plagioclase-bearing zones. This illustrates a certain amount of diffusion and the non-compatibility of grossularite with the approaching plagioclase-augite-leucite assemblage.

The other assemblages are partly connected by transitions, which could not all be shown in the figure. This counts especially for the plagioclase-

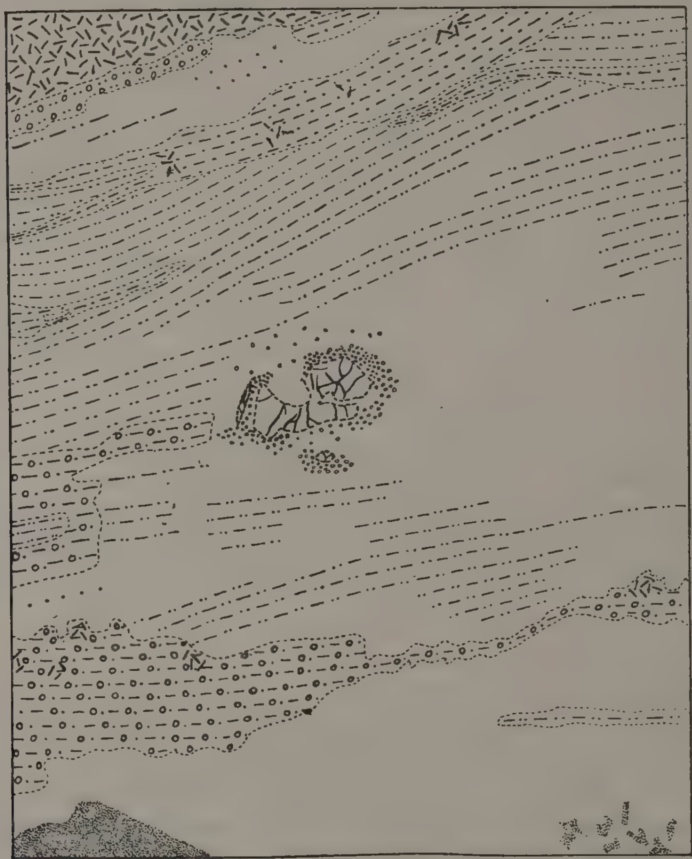


Fig. 4. Enl. × 6.
A partly altered crystal of grossularite in the centre of the figure.

augite assemblages with and without wollastonite and the wollastonite-augite assemblages with and without plagioclase.

In the zones or bands in which plagioclase is found with wollastonite and augite, the plagioclase, which has the composition of bytownite to bytownite-anorthite, is often elongated in the direction of the parallel structure. Crystals with a length of 1.5 mm and measuring only 0.15 mm across have been observed, they do not show good crystal boundaries and are crowded with inclusions consisting of the two other minerals of the assemblage, augite and wollastonite. A moderate amount of inclusions of liquid or gas is also found. The augite appears in various shades of greenish, brownish and purplish grey tints; it is up to 0.2 mm in diameter

but the crystals are mostly much smaller and reach down to about 0.005 mm in size.

The zone in which plagioclase and augite are the main constituents locally contains interstitional glass-bearing spots with microlites of potash-rich felspar. Calcite, pyrrhotite and titanite are also found. The size of the plagioclase, which is very basic, ranges up from about 0.02 mm in fine grained parts to 0.5 mm in crystals, which are crowded with inclusions. The augite also varies much in size; skeletons of this mineral reach up to 0.8 mm in diameter. The crystals of plagioclase partly show a zonal structure with more acid rims. Near the glass-bearing spots they are often partly surrounded by potash-rich felspar.

Of the leucite-bearing assemblages, in which plagioclase and augite are the principal minerals, that to the left of the garnet crystal shows the coarsest grain. Plagioclase crystals up to 2 mm in size have been observed while the size of the largest crystal in the lower part of the figure is 1 mm. The plagioclase is bytownite to bytownite-anorthite with up to 93 % An, sometimes with a narrow more acid rim. Oscillatory zoning was observed in some crystals. Near the glass-bearing parts, which are found locally, the plagioclase is often partly surrounded by potash-rich felspar, which also forms microlites in these glass-bearing parts. The augite, which shows various shades of greenish, brownish and purplish grey tints, reaches up to 0.4 mm in size but generally the size is much smaller. Many augite crystals are enclosed in plagioclase. The leucite shows optical anomalies, crystal faces are absent and it encloses numerous crystals of augite. Its growth has been hindered by the plagioclase, which often shows good crystal boundaries where it borders upon leucite. Other constituents of the assemblage are titanite and interstitional calcite; the titanite is partly more or less idiomorphic and reaches up to 0.1 mm in size. Pyrrhotite and magnetite, both up to 0.1 mm in diameter, are rarely found. Wollastonite occurs enclosed in plagioclase and also in leucite, it partly forms needle-shaped crystals, which often show a radiate arrangement.

The narrow zone, which borders upon the zone of trachyte in the upper left corner of the figure, mainly consists of leucite and augite. The diameter of the leucite is up to 0.5 mm, it encloses numerous crystals of augite and some small crystals of titanite. Calcite is found in spots and small veins. The adjacent trachyte is strongly vesicular; it contains phenocrysts of potash-rich felspar, which are up to 0.4 mm in length. The glass-bearing groundmass contains numerous lath-shaped crystals and small microlites of potash-rich felspar, which reach up to 0.15 mm in length, and small crystals of an amphibole with affinities to katophorite of which the size does not exceed 0.05 mm.

From the above description it is evident that a stage of metamorphism during which the limestone was altered into wollastonite and garnet was followed by a stage during which the plagioclase-, augite- and leucite-bearing assemblages were formed in more or less parallel zones. This stage

was interrupted when the limestone block was ejected and the still liquid part of the magmatic substances consolidated as glass.

Fig. 5.

The main part of the figure shows a wollastonite-rock in which various mineral assemblages are found in more or less parallel zones. A parallel structure is also more or less distinctly shown in the greater part of the plagioclase-augite zones, especially in the left half of the figure where the signature is parallel to the structure. Bifurcating zones of trachytic and leucitephonolitic composition are partly parallel and partly oblique to the main structure. Limestone appears in the left lower corner of the figure.

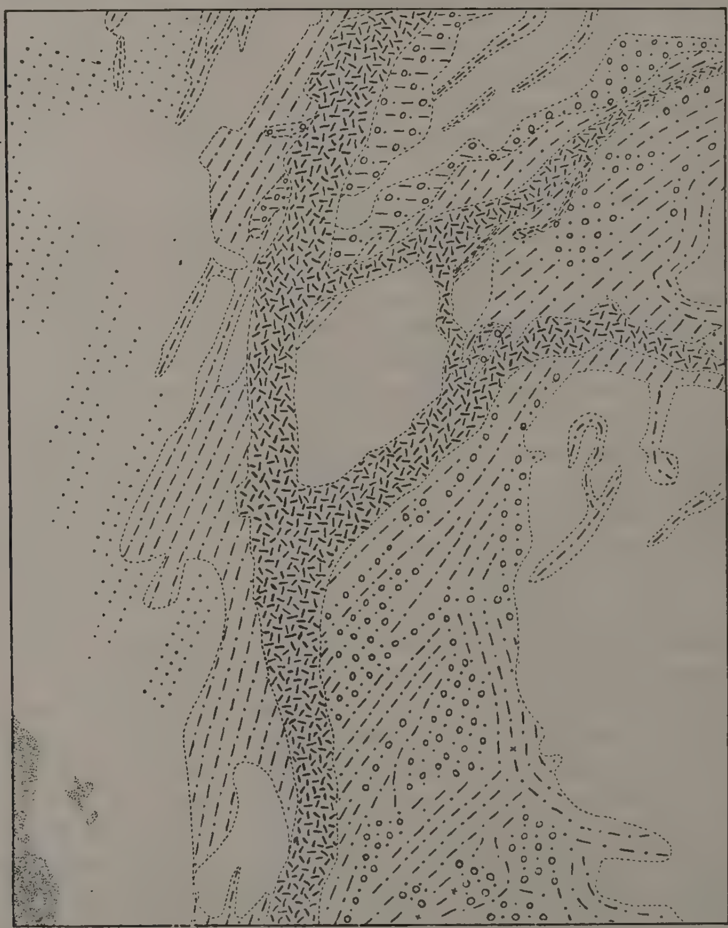


Fig. 5. Enl. × 6.

The wollastonite-rich parts consist nearly entirely of this mineral, it reaches up to 1 mm in size but the crystals are mostly much smaller (about 0.1—0.2 mm). Pyrrhotite and magnetite are both present in subordinate

amount. Gehlenite has not been observed. The crystals of augite, which are arranged in more or less parallel zones in the wollastonite-rich parts, are up to 0.2 mm in size. Their colour varies often in the same crystal from greyish brown to nearly colourless or light greyish brown; greyish green tints have also been observed.

The zones in which basic plagioclase and augite are the main constituents show a varying grain size. For the greater part the plagioclase occurs in minute crystals of which the size is about 0.01—0.03 mm. Polysynthetic twinning is often observed. Plagioclase-rich and augite-rich parts often alternate in more or less parallel narrow bands and the longer axes of the augite crystals show a tendency to be arranged parallel to these bands. The augite crystals range up in size from 0.005 mm in the smallest grains to 1.2 mm in the largest observed skeleton. The large crystals sometimes consist of separate parts with simultaneous extinction. Their colour is brownish to greenish grey, purplish grey tints are more particularly found in the right part of the figure between the spots which are rich in leucite. Especially in this part a residual cement, consisting of leucite, is clearly developed between the plagioclase grains. Many, mostly irregular, crystals of titanite occur in the assemblage; other constituents are calcite and some grains of iron ore. Flakes of brown biotite were observed locally. A larger size of the plagioclase (up to 0.5 mm) is found at some places near the border of the fragments of wollastonite-rock and the colour of some of the augites is a little darker there. The largest size of the plagioclase crystals (up to 1 mm) was observed in the upper left part of the figure where the characteristics of the assemblage are similar to those in figures 1 and 2. In irregular spots in wollastonite, which are shown in the right middle part of the figure, the main constituent is greenish to brownish grey augite of varying size. Larger skeletons reach up to 1 mm in diameter. Plagioclase, calcite and iron ore are found in subordinate amount; the size of the plagioclase crystals does not exceed 0.1 mm.

In the leucite-rich assemblages leucite and augite, whether or not with plagioclase, are the main constituents. They mostly form spots in the plagioclase-augite mixture in the right part of the figure. These spots, which are roughly shown in the figure, range down to a very small size and pass into the leucite cement between the grains of the adjacent plagioclase-augite mixture, which is also found enclosed in the spots. The spots contain minute grains and larger prismatic and irregular crsytals of greenish grey and purplish grey augite, which reach up to 0.75 mm in size. The leucite forms a base in which the other minerals are embedded and the boundaries between the individual crystals are not distinct. Besides the leucite-rich assemblages, in which plagioclase grains similar to those of the adjacent mixture are enclosed, there are also assemblages in which the plagioclase is partly of larger size. These parts are shown with a different signature in the upper part of the figure. The plagioclase is up to 0.2 mm in diameter,

it is of very basic composition and sometimes shows a zonal structure with a more acid rim. Good crystal boundaries are often found. The grey to greenish grey augite reaches up to 0.5 mm in size, it often shows good crystal boundaries but it also forms skeletons of which isolated parts extinguish simultaneously. Other constituents are calcite, titanite, some minute crystals of iron ore and an interstitional glass-bearing substance similar to that of the leucitephonolite, which will be described below.

In the bifurcating zones or veins of leucitephonolitic composition the leucite crystals are mainly arranged along the border, which may for some distance consist of a continuous band of leucite; these bands are not shown in the figure by a different signature. Some crystals are found as small phenocrysts, which are entirely surrounded by the glass-bearing groundmass of the inner part of the zones. Leucite is lacking in the narrower, trachytic, parts of the glass-bearing zones in the right upper part of the figure. The leucite crystals are from 0.2 to 0.75 mm in diameter. They show optical anomalies, they partly contain narrow veins of calcite and they may enclose small plagioclase grains and augite crystals of different size. The greenish grey augite is also found in separate crystals of phenocrystic size. The groundmass contains potash-rich felspar, brownish amphibole, light greenish to nearly colourless augite and glass. Basic plagioclase, titanite and iron ore are present in small amount. Some leucite crystals have about the same size as the largest constituents of the groundmass, which is strongly vesicular;the vesicles are partly filled with calcite. The lath-shaped crystals of potash-rich felspar show parallel or nearly parallel extinction; they are up to 0.2 mm in length. Small microlites with a length of about 0.01—0.03 mm are numerous. The larger crystals may surround a small core of basic plagioclase, which reaches up to about 0.03 mm in diameter. Small plagioclase grains without a rim of potash-rich felspar are also found. The amphibole crystals have an irregular or more or less prismatic shape and reach up to 0.1 mm in length. Their colour is mostly brown but various shades of brown, bluish purple and green tints are sometimes found in the same crystal. The low double refraction, the negative elongation and the large extinction angles (up to 35°) point to an alkali-amphibole with affinities to katophorite. The interstitional glass shows slight devitrifcation.

The continuous bands of leucite crystals along parts of the border of the glass-bearing zones may resemble the holocrystalline leucite-rich assemblages outside these zones, which are shown by a different signature in the figure. The glass-bearing zones contain potash-rich felspar and amphibole. Potash-rich felspar is found in holocrystalline assemblages in other parts of the rock (comp. figs. 3 and 13) but amphibole appears in the glass-bearing parts only. This indicates that the stability field of amphibole was reached shortly before the final consolidation of the last residual liquid.

Fig. 6.

Limestone is absent in this figure. Large parts of the wollastonite-rock contain numerous crystals of greenish grey augite, which are often accompanied by calcite and small amounts of plagioclase or potash-rich

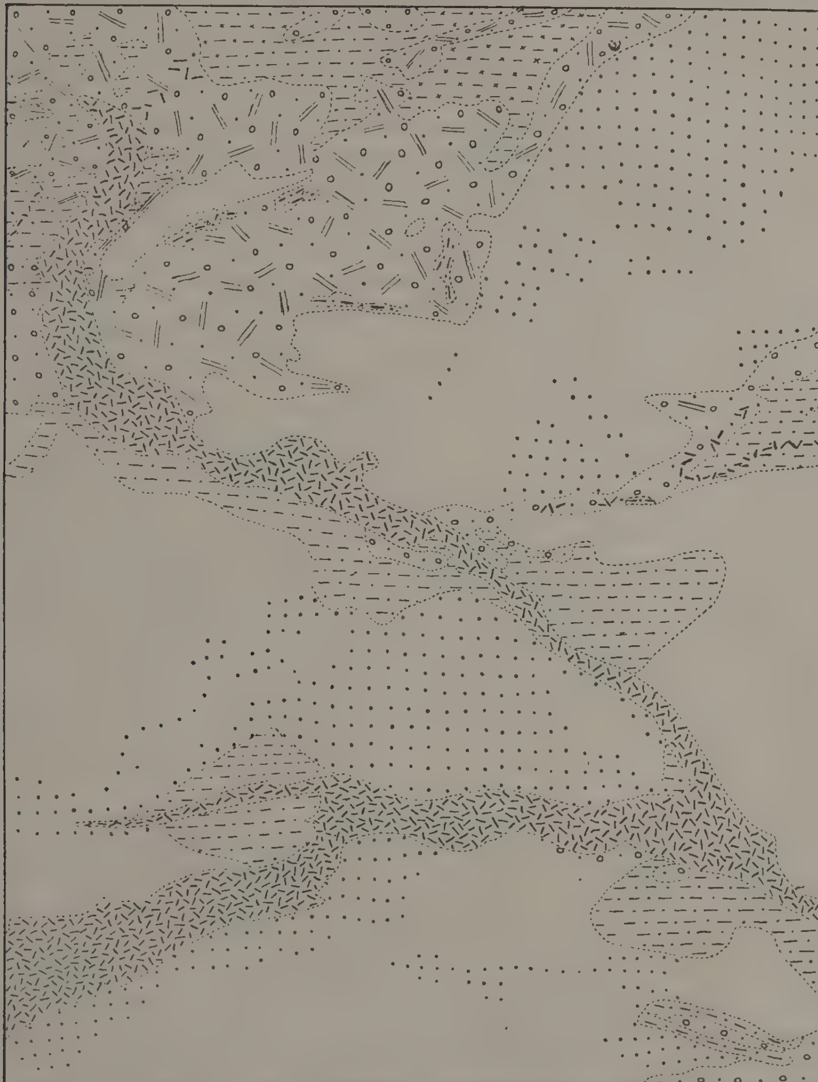


Fig. 6. Enl. × 6.

felspar. These parts are roughly shown in the figure. The larger crystals of augite are skeletons, which reach up to 0.6 mm in size. Pyrrhotite is a rare constituent of the wollastonite-rock and gehlenite seems to be absent.

The assemblages in which plagioclase and augite or plagioclase and

biotite are the main constituents resemble those in fig. 3. Locally augite and biotite are found together but mostly they are separated in space. The average size of the plagioclase and biotite crystals is about 0.01 mm, the largest crystals are up to 0.05 mm in diameter. Skeletons of augite may reach up to 0.7 mm and irregular crystals of titanite may reach up to 0.1 mm in size. Small crystals of magnetite and pyrrhotite occur in small amount. Leucite and potash-rich felspar are found as an interstitional cement in these assemblages.

The two last-mentioned minerals are found in crystals of larger size in the assemblages in which leucite and augite or leucite, potash-rich felspar and augite are the main constituents. The average grain size varies from about 0.06 to 0.4 mm but locally the crystals of potash-rich felspar reach up to 1.5 mm in length and the same size is reached by some larger skeletons of augite. Smaller crystals of augite often show good crystal boundaries, their colour mostly varies from greenish to purplish grey. Some crystals are almost colourless. A greenish rim surrounding a colourless core with smaller extinction angle is sometimes observed. Calcite, titanite and iron ore are constituents of minor importance while wollastonite is found locally, especially near the boundary of the wollastonite-rock.

The leucitephonolite and trachyte are strongly vesicular and resemble those in fig. 5. The leucite crystals are up to 0.9 mm in diameter, are mainly found along the border of the zones or veins (fig. 14) and are lacking in parts of them; these parts have a trachytic composition. Small phenocrysts of potash-rich felspar, which reach up to 0.25 mm in length are observed locally. The lath-shaped crystals of potash-rich felspar in the groundmass sometimes have minute cores of basic plagioclase. The amphibole with affinities to kataphorite shows different colours, often with the lightest colour in the marginal part. The numerous vesicles are partly empty and partly contain calcite.

Fig. 7.

The fragments of wollastonite-rock show similar characteristics as those in fig. 6. In the left lower part of the figure the limestone is not yet altered entirely into wollastonite (with some gehlenite and pyrrhotite). Larger crystals of wollastonite and gehlenite reach up to 0.8 mm in size but the grain size is mostly much smaller (about 0.1 mm). A fine grained gehlenite-rich zone with an average grain size of about 0.1 mm is found between the partly altered limestone and the surrounding plagioclase-augite zone. This gehlenite-rich zone is absent where the limestone is entirely altered. The gehlenite-rich zone also contains wollastonite, garnet, augite and calcite; its mineralogical composition resembles that in the left lower part of fig. 3 where its presence is also connected with a core of limestone in the lime silicate assemblage.

The assemblages in which plagioclase and augite or plagioclase and biotite are the main constituents resemble those of fig. 3. They contain

numerous leucite-bearing spots (fig. 12), which are roughly shown in the sketch.

Assemblages in which leucite and augite, potash-rich felspar and augite,

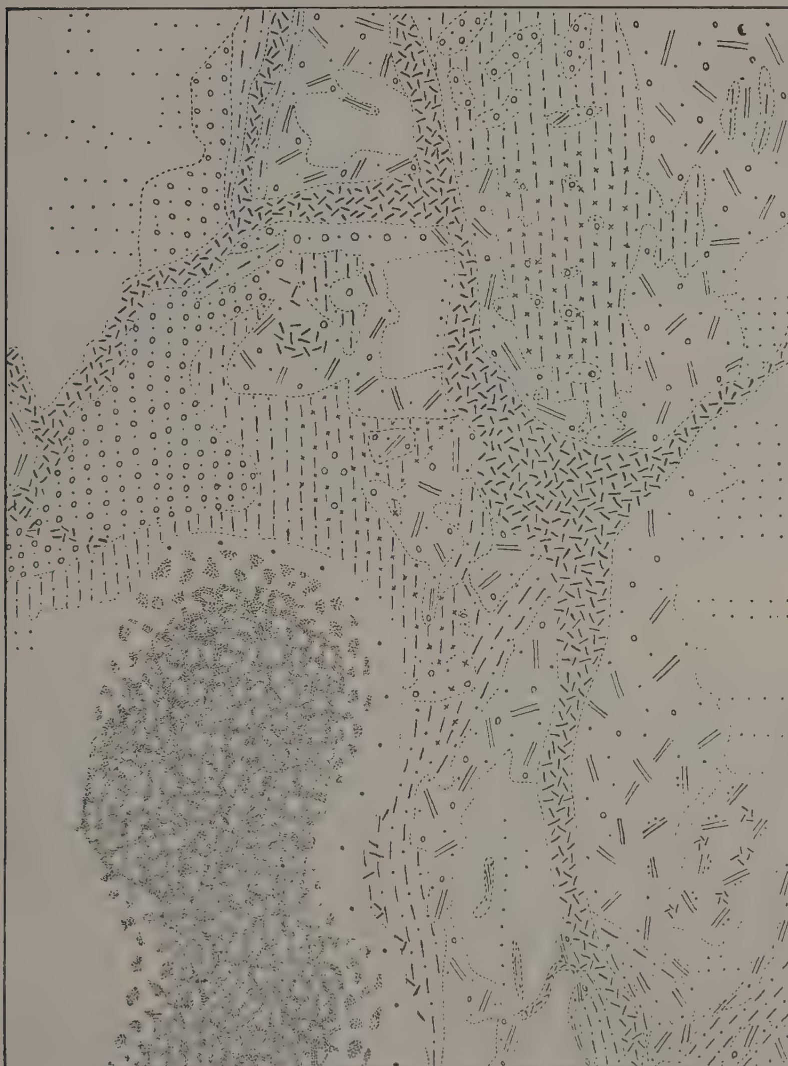


Fig. 7. Enl. × 6.

or leucite, potash-rich felspar and augite are the main constituents cover a large part of the figure. The grain size is varying; the crystals of potash-rich felspar vary from 0.06 mm to 1 mm in size, some crystals of leucite reach a diameter of more than 1 mm and the size of the augite crystals ranges up from less than 0.02 mm in the small grains to 1.5 mm in the

Fig. 8.

Beginning alteration of limestone into wollastonite and gehlenite as shown in the upper part of fig. 2. Slightly recrystallized limestone in the central part of the figure. Dark spots and zones with small crystals of wollastonite and gehlenite show the first stage of alteration. More crystals of wollastonite and gehlenite in the upper and lower parts of the figure. Ordinary light. $\times$ 50.

Fig. 9.

Protruding sector of the outer zone around a limestone lenticle. The main constituents of the core in the left lower part of the figure are gehlenite, wollastonite, augite and iron ore. The augite content increases towards the margin, the outer rim consists of augite and calcite. Augite crystals of the plagioclase-augite mixture (comp. fig. 3) in the upper and right part of the figure partly lie inside the outer rim. Ordinary light. $\times$ 24.

Fig. 10.

Strongly vesicular, pyroxene-rich section in the right upper part of fig. 3 with small crystals and larger skeletons of pyroxene in a slightly devitrified glass. The vesicles are irregular in shape and mostly empty. Ordinary light. $\times$ 46.

Fig. 11.

Zonal arrangement of different mineral assemblages in the right upper part of fig. 1. Wollastonite in the left part of the figure, farther to the right plagioclase and augite (two white plagioclase crystals are clearly visible). In the right half of the figure: leucite, augite and wollastonite with some calcite. Nicols crossed. $\times$ 18.

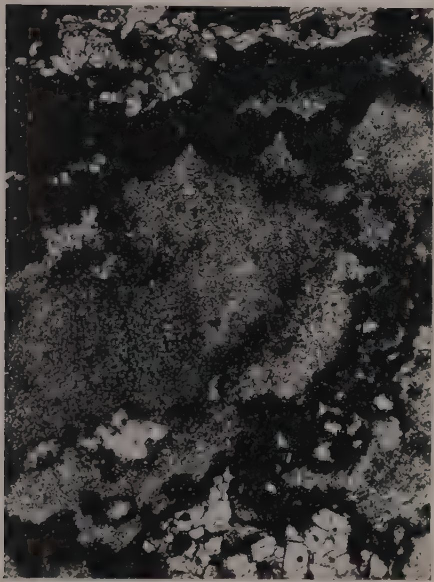


Fig. 8.

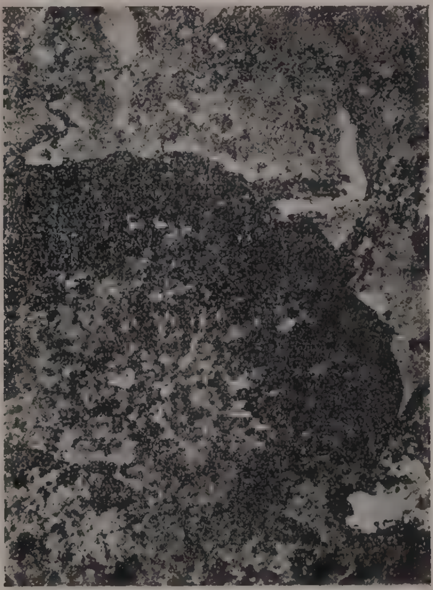


Fig. 9.

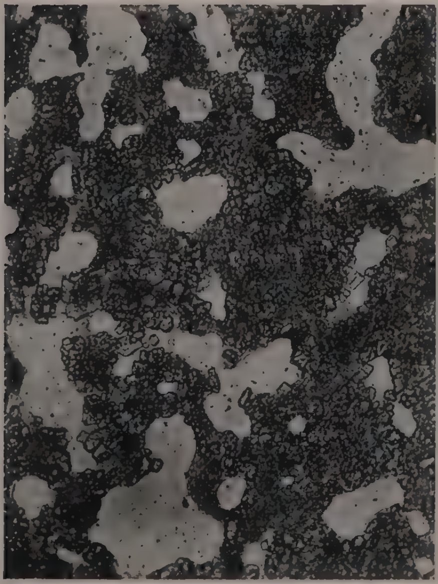


Fig. 10.

Fig. 11.

Fig. 12.

Fig. 13.

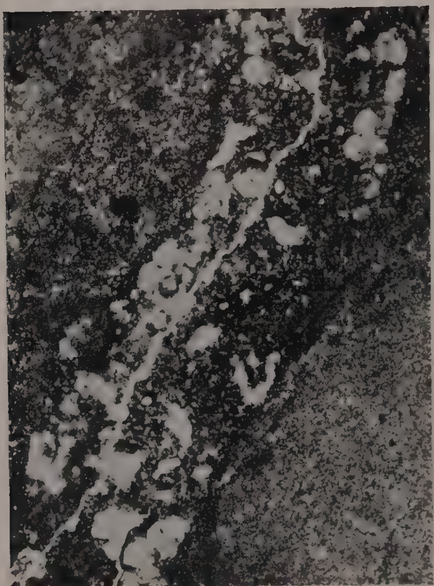


Fig. 14.

Fig. 15.

Fig. 12.

Leucite-rich spots in a fine-grained mixture of plagioclase and augite or plagioclase and biotite, as shown in the upper part of fig. 7. The spots contain leucite, augite and sometimes biotite. Ordinary light. × 29.

Fig. 13.

Spots containing potash-rich felspar, augite and sometimes biotite in a fine-grained mixture of plagioclase and biotite, as shown in the left part of fig. 3. Ordinary light. × 18.

Fig. 14.

Vesicular leucitephonolite in the lower half of fig. 6. The leucitephonolite is seen from the left lower to the right upper part of the figure. The vesicles are irregular in shape and mostly empty. The leucite crystals are found in the marginal parts of the vein; near the left border they are especially numerous while near the right border they are only found in the upper and lower parts of the figure. The crystals have euhedral and rounded forms and often carry more or less concentric inclusions. The leucitephonolite is bounded on both sides by wollastonite-rock, which is poor in augite in the right part and rich in augite in the left part of the figure. Ordinary light. × 16.

Fig. 15.

Trachytic zone with phenocrysts of potash-rich felspar in a glass-rich vesicular base. The zone pinches out near the right lower corner of the figure. To the right of the trachyte is a zone consisting of gehlenite and wollastonite, which, outside the figure, borders upon a limestone lenticle. To the left of the trachyte is a zone consisting of plagioclase and augite with interstitional glass-rich spots containing microlites of potash-rich felspar. Farther to the left, outside the figure, the main constituents of the alternating zones are wollastonite and augite or plagioclase and augite. Nicols crossed. × 14.

largest observed skeleton. The colour is somewhat varying from greenish grey to nearly colourless, while a greyish to brownish green colour seems to prevail in the leucite-rich assemblages. Titanite and iron ore are minor constituents; with numerous small crystals of augite they are enclosed in potash-rich felspar and in leucite. Calcite and wollastonite occur locally. Glass-bearing spots with microlites of potash-rich felspar are exceptionally found. In the right lower part of the figure brown glass and calcite are found in a mixture of augite, wollastonite and potash-rich felspar. Most of the glass is found in the bifurcating zones of leucitephonolitic and trachytic composition, which are similar to those of figures 5 and 6.

Replacement of the limestone and production ol alkali rocks.

The figures and petrographic descriptions in the foregoing pages show that the limestone has for the greater part been converted into silicates. The lenticles and strings, which have not been converted, consist of a rather pure calcium carbonate. This indicates that permeation of the limestone by magmatic solutions was the principal cause of the metamorphic changes. It seems that fissuring has facilitated the permeation to a certain degree and a more forcible injection process may also have contributed to the invasion of magmatic substances.

A succession of crystallization of the different minerals can be noted but their periods of formation overlap. Later minerals may replace the earlier ones. It is often clear, that the succession of the principal constituents began by wollastonite and gehlenite, that augite and plagioclase are later and that potash-rich felspar and leucite are still later. There is a tendency for the potash-rich felspar to occur as mantles about plagioclase as is the case in alkali rocks from some Javanese and other volcanoes [1]). Gehlenite, wollastonite and augite may have continued to develop until a late stage of the metamorphic process, especially where they are found around a core of limestone (figures 3, 7, 9).

The mineralogical composition of the metamorphic rock shows that many substances — chiefly silica, alumina and alkalies — have been introduced into the limestone. Carrying away of calcium oxide and liberation of carbon dioxide could compensate this addition of substances. Evidence of inoculation of the surrounding magma by introduction of calcium oxide may be found in the more or less rounded glass-rich and strongly vesicular sections in the upper part of fig. 3, which occur at the surface of the limestone block and which are very rich in crystals of light-coloured pyroxene. These crystals lie in a slightly devitrified brownish to colourless glass (fig. 10). Liberated carbon dioxide could assist in decreasing the

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[1]) J. P. IDDINGS and E. W. MORLEY. Contribution to the petrography of Java and Celebes. Journ. of Geol., 23, 1915, p. 231—245. N. L. BOWEN. The evolution of the igneous rocks, 1928, p. 229 and 250.

viscosity and may explain the abnormal development of vesicles and the growth of comparatively large pyroxene skeletons (fig. 10).

Evidence of a special phase of crystallization differentiation may be found in the spots which contain potash-rich felspar or (and) leucite (figs. 3, 7, 12 and 13). There is a sharp break in composition and grain size between the spots and the surrounding mineral assemblage which is rich in basic plagioclase. Volatile constituents, which carried the alkalies with them may have concentrated in the spots where they decreased the viscosity of the residual liquid. The abundance of volatile constituents is particularly well illustrated by the strongly vesicular character of the phonolite, trachyte and trachyandesite. Limited differentiation may, also have influenced the zonal arrangement of different mineral assemblages.

The connection between the undersaturated rocks and the typical Merapi lava has not been observed as this lava was not found adherent to the metamorphic limestone. It is, however, clear that in a volcano, which erupts pyroxeneandesites, a great variety of alkali rocks has been produced on a small scale in association with limestone. This production was limited in space and time and it raises the question whether a large scale production of alkali rocks by reaction of igneous magma with limestone, under more favourable conditions, was realized in extinct volcanoes near the north coast of the eastern part of Java where the volcanic rocks are largely alkali-rich [1]).

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[1]) H. A. BROUWER. loc. cit.

Physics. — *Note on the theory of vector wave fields.* By L. J. F. BROER (ZEEMAN-laboratory of the University of Amsterdam) and A. PAIS (Instituut voor Theoretische Natuurkunde Rijks-Universiteit Utrecht). (Communicated by Prof. J. D. VAN DER WAALS JR.)

(Communicated at the meeting of October 27, 1945.)

Summary.

It is remarked that FERMI's method of treating the electromagnetic field also can be applied to the classical, but not to the quantum theory of the vector meson field. The rôle of the LORENTZ condition is examined. The connection between the electromagnetic and the (neutral) vector meson field is discussed.

§ 1. *Introduction.* Whereas, from a classical point of view, the electromagnetic field equations may be considered as the equations for a neutral vector meson field with vanishing meson rest mass, it is well known, however, that the quantized MAXWELL equations cannot be obtained by letting

$$\varkappa = \frac{mc}{\hbar} \rightarrow 0$$

in the PROCA [1]) equations *). This "lack of continuity" can be traced back to the following two causes:

a. In the customary treatment of the MAXWELL as well as of the PROCA field, the canonically conjugate of the scalar potential, (i.e. the time component of the four-vector potential) vanishes. However, in the latter case the scalar potential can be treated as "derived variable" [2]) i.e., by expressing it in terms of (spatial derivatives of) canonical variables, it can be eliminated from the Hamiltonian of the field; this is impossible in the electromagnetic case.

b. The different rôle of the so-called LORENTZ condition in both cases: This condition is introduced in the MAXWELL case for the sake of convenience, it having no bearing on the field strengths and field energy. In dealing with the vector meson field it is, however, again in the customary treatment, a direct consequence of the field equations themselves.

As to the first point, it has been shown by FERMI [3]) that the canonically conjugate of the scalar potential in the MAXWELL case need not be zero, provided one starts from a suitable Lagrangian of the field. In this note

*) In this note we shall understand by PROCA equations the field equations for a *real* vector meson field.

FERMI's method [4]) will be applied to the PROCA field from which it will be shown (§ 2):

a'. That neither in the MAXWELL nor in the PROCA case the canonically conjugate of the scalar potential vanishes.

b'. That in both cases the LORENTZ condition can be treated as an accessory condition.

Still it will appear that it is even now impossible to treat the quantized MAXWELL equations as a special case of those of PROCA by taking the limit for $\varkappa = 0$. The reason for this will be seen to be the different consequences which the occurrence of the LORENTZ condition has in the *quantum* theory of the corresponding wave fields. In fact, it turns out (§ 4) that the limiting transition $\varkappa \to 0$ is singular in the same way as a LORENTZ transformation for $v/c \to 1$.

The possibility of the treatment of the LORENTZ condition as an accessory one being given, one might ask whether it is necessary to introduce it at all. If one would do without it, however, the field energy appears not to be positive definite. This has already been stated by FIERZ [5]) who has, moreover, pointed out that as a consequence of the LORENTZ condition only spin 1 particles occur upon field quantization, while a theory without this condition would also yield particles with spin zero. As it is, furthermore, just these latter which make the energy not positive definite, it is clear that one cannot do without the LORENTZ condition, whatever its way of occurrence in the theory may be.

The present paper is mainly of a methodical character. Though some points raised here cannot be claimed to be new, the lack of a general survey of the problems discussed here let it seem justified to include them in this note.

§ 2. *Variation principles.* The following notations are used:

$$a_{[\lambda\mu]} \equiv a_{\lambda\mu} - a_{\mu\lambda},\ \partial_\mu = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t},\ (c = 1).$$

Greek indices run from 1 to 4; summation signs are suppressed. Co- and contravariant tensor components are connected in the usual way.

The MAXWELL equations can be written as:

$$F_{\lambda\mu} = f_{[\lambda\mu]};\ f_{\lambda\mu} \equiv \partial_\lambda A_\mu, \quad \quad (1a)$$

$$\partial_\mu F^{\mu\nu} = 0, \quad \quad (2a)$$

where A_λ is the four-vector potential. Using the same notation for the PROCA field, we have:

$$F_{\lambda\mu} = f_{[\lambda\mu]};\ f_{\lambda\mu} \equiv \partial_\lambda A_\mu, \quad \quad (1b)$$

$$\partial_\mu F^{\mu\nu} = \varkappa^2 A^\nu. \quad \quad (2b)$$

Elimination of the field tensors $F_{\mu\nu}$ yields the second order differential equations:

$$\square A^\nu - \partial^\nu \partial_\mu A^\mu = 0, \ \square \equiv \partial_\lambda \partial^\lambda, \text{ MAXWELL} \quad \ldots \quad (3a)$$

$$\square A^\nu - \partial^\nu \partial_\mu A^\mu = \varkappa^2 A^\nu, \text{ PROCA.} \quad \ldots \quad (3b)$$

If we next introduce, in the MAXWELL case, the accessory condition

$$\partial_\mu A^\mu = 0, \quad \ldots \quad (4)$$

which is known as the LORENTZ condition, (3a) becomes:

$$\square A^\nu = 0. \quad \ldots \quad (5a)$$

In the PROCA case, on the other hand, (4) is a consequence of (2b) on account of the antisymmetry of $F_{\mu\nu}$. Thus (3b) becomes:

$$\square A^\nu = \varkappa^2 A^\nu. \quad \ldots \quad (5b)$$

Considering (1a, b) as defining $F_{\mu\nu}$, the equations (2a, b) can be derived from the variation principle:

$$\delta \int \mathfrak{L}\, dv\, dt = 0, \quad \ldots \quad (6)$$

with

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\varkappa^2}{2} A_\mu A^\mu; \quad \ldots \quad (7)$$

in the MAXWELL case $\varkappa = 0$ of course. The quantization of the fields starting from (6) and (7) is well known and needs no recapitulation here. We only remind that the canonical conjugate of A_4 vanishes as $\partial_4 A_4$ does not occur in (7). This leads to some complications in the MAXWELL but not in the PROCA case as A_4 can there be expressed in terms of the canonical conjugates of A_1, A_2 and A_3 with the aid of the fourth equation (2b).

FERMI's method now consists in starting from (6) but taking *)

$$\mathfrak{L} = -\tfrac{1}{2} f_{\mu\nu} f^{\mu\nu} - \frac{\varkappa^2}{2} A_\mu A^\mu, \quad \ldots \quad (8)$$

where $f_{\mu\nu}$ is defined by (1a, b). It is easily seen that the field equations directly derived from (8) are (5a) and (5b) respectively. Thus if we start from (8), it is, at this stage, not necessary to use (4) to establish these field equations. In point of fact, (4) cannot now be derived from the field equations and the definition of $f_{\mu\nu}$, because the latter is not antisymmetrical; and this is true whether $\varkappa$ is or is not equal to zero. It may be noted that instead of (4) we now get

$$(\square - \varkappa^2) S = 0; \ S \equiv \partial_\nu A^\nu. \quad \ldots \quad (4')$$

*) An objection of principle which may be raised against the use of (8) with $\varkappa = 0$ is that the Lagrangian is not gauge invariant. (If $\varkappa \neq 0$ there is no gauge group of course.)

However, as will appear below, we do need (4) in the MAXWELL as well as in the PROCA case in order to obtain a positive definite field energy. But, if we use (8), (4) should now be considered as an accessory condition in both cases. That, using (8) in stead of (7), the accessory condition is necessary from a physical point of view is connected with the circumstance that $f_{\mu\nu}$ is not antisymmetrical here. This ensues that a choice must be made from the solutions of (5) so as to get the correct value of the energy. This choice will appear to be (4).

§ 3. FOURIER *development; energy of the field.* Assuming the system to be enclosed in a cube of volume 1 with periodicity conditions on its boundary, and introducing the vector notation

$$A_\mu = \vec{A}, -B,$$

$\vec{A}$ and B can be developed in a FOURIER series in the following way:

$$\left.\begin{aligned} \vec{A} &= \sum_{j=0,1,2} \sum_{\vec{k}} \vec{e}_{j\vec{k}} \left[A(j,\vec{k})\, e^{i(\vec{k}\,\vec{x})} + A^\dagger(j,\vec{k})\, e^{-i(\vec{k}\,\vec{x})}\right], \\ B &= \sum_{\vec{k}} \left[B(\vec{k})\, e^{i(\vec{k}\,\vec{x})} + B^\dagger(k)\, e^{-i(\vec{k}\,\vec{x})}\right], \\ \vec{e}_{0\vec{k}} &= \frac{\vec{k}}{|\vec{k}|}, \quad \vec{e}_{j\vec{k}}\, \vec{e}_{j'\vec{k}} = \delta_{jj'}, \end{aligned}\right\} \quad . \quad . \quad (9)$$

i.e. $j = 0$ denotes longitudinal, $j = 1,2$ transverse waves. Substituting (9) in (8) we get *):

$$\left.\begin{aligned} L = \int \mathfrak{L}\, dv = \sum_{j=0,1,2} \sum_{\vec{k}} &\left[\dot{A}^\dagger(j,\vec{k})\, \dot{A}(j,\vec{k}) - \bar{\nu}_k^2 A^\dagger(j,\vec{k})\, A(j,\vec{k})\right] \\ &- \sum_{\vec{k}} \left[\dot{B}^\dagger(\vec{k})\, B(\vec{k}) - \bar{\nu}_k^2 B^\dagger(\vec{k})\, B(\vec{k})\right], \\ \bar{\nu}_k^2 = \nu_k^2 + \varkappa^2, \quad \nu_k &= |\vec{k}|. \end{aligned}\right\} \quad . \quad . \quad (10)$$

In the usual way we obtain from (10) the equations of motion for the amplitudes:

$$\ddot{A}(j,\vec{k}) + \bar{\nu}_k^2 A(j,\vec{k}) = 0, \quad \ddot{B}(\vec{k}) + \bar{\nu}_k^2 \vec{B}(k) = 0 \quad . \quad . \quad . \quad (11)$$

(11) can also be obtained by substituting (9) in (5a, b); if $\varkappa = 0$, $\bar{\nu}_k = \nu_k$.

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*) In expressions like (10) the right member should be hermitized. For simplicity we write (10) as it stands.

The theory is therefore consistent so far. But difficulties arise because the Hamiltonian derived from (10):

$$\left.\begin{aligned} H = \sum_j \sum_{\vec{k}} [\dot{A}^{\dagger}(j,\vec{k})\,\dot{A}(j,\vec{k}) + \bar{\nu}_k^2\,A^{\dagger}(j,\vec{k})\,A(j,\vec{k}) - \\ - \sum_k [\dot{B}^{\dagger}(\vec{k})\,\dot{B}(\vec{k}) + \bar{\nu}_k^2\,B^{\dagger}(\vec{k})\,B(\vec{k})], \end{aligned}\right\} \quad . \quad . \quad (12)$$

is not positive definite. And it is here that (4) plays an essential rôle. In amplitudinal form, (4) can be written as

$$\dot{B}(\vec{k}) = -i\,\nu_k\,A(0,\vec{k}),\ \ \dot{B}^{\dagger}(\vec{k}) = i\,\nu_k\,A^{\dagger}(0,\vec{k}).$$

This can with the aid of the equations of motion be brought in the form:

$$\bar{\nu}_k\,B(\vec{k}) = \nu_k\,A(0,\vec{k}),\ \ \bar{\nu}_k\,B^{\dagger}(\vec{k}) = \nu_k\,A^{\dagger}(0,\vec{k}). \quad . \quad . \quad . \quad (13)$$

Now there are two ways of introducing an accessory condition in classical theory which we shall discuss successively with regard to their bearing on our problem.

The first method, which is not applicable generally, consists of looking for solutions of the restricted problem (i.e. the problem involving the condition) under those of the general problem (not involving the condition). In our case this means that we have to look for solutions of (5*b*) which satisfy (4). $S \equiv 0$ obviously is a solution of (4′) *). It is characterized by the boundary conditions: three dimensional periodicity, S and $\dot{S} = 0$ for $t = t_0$. As these conditions are fulfilled for a class of solutions of (5*b*), the first method is valid here. This result is independent of the value of $\varkappa$. Thus, both if $\varkappa = 0$ and $\varkappa \neq 0$ the LORENTZ condition $S \equiv 0$ can be replaced by $S = \dot{S} = 0$ at a given time, in vacuum as well as with a charge current density.

The second method, closely related to LAGRANGE's parameter method, simply consists in substituting (13) into (12) which yields the positive definite form:

$$\begin{aligned} H = \sum_{j=1,2} \sum_{\vec{k}} [\dot{A}^{\dagger}(j,\vec{k})\,A(j,\vec{k}) + \bar{\nu}_k^2\,A^{\dagger}(j,\vec{k})\,A(j,\vec{k})] + \\ + \sum_{\vec{k}} \frac{\varkappa^2}{\bar{\nu}_k^2}\,[\dot{A}^{\dagger}(0,\vec{k})\,\dot{A}(0,\vec{k}) + \bar{\nu}_k^2\,A^{\dagger}(0,\vec{k})\,A(0,\vec{k})] \quad . \quad (14) \end{aligned}$$

which holds for any $\varkappa$; thus the second method too applies to both cases. An equivalent, but more direct, way to obtain (14) is to start from the Lagrangian (7) instead of (8). If $\varkappa = 0$ we only get a contribution of the transverse waves, as it should be.

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*) It is interesting, though not essential for the present purpose, to note that (4′) remains valid when the charge current density is introduced on the right hand side of (5*b*).

§ 4. *Quantization of the field.* According to (10), the canonical conjugate of $A(j, \vec{k})$ is $\dot{A}^{\dagger}(j, \vec{k})$, that of $B(\vec{k})$ is $-\dot{B}^{\dagger}(\vec{k})$, etc. In order to quantize the field we have therefore to postulate the relations:

$$\left.\begin{aligned} &[\dot{A}^{\dagger}(j, \vec{k}), A(j', \vec{k}')] = -i\hbar\,\delta_{jj'}\,\delta_{\vec{k}\vec{k}'}, \\ &[\dot{B}^{\dagger}(\vec{k}), B(\vec{k}')] = i\hbar\,\delta_{\vec{k}\vec{k}'}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

all other pairs commuting. In the usual way we find from (15) that the Hamiltonian takes the form, (the notation will be obvious):

$$H = \sum_{\vec{k}} \hbar\,\bar{\nu}_k\,[N_A(1, \vec{k}) + N_A(2, \vec{k}) + N_A(0, \vec{k}) - N_B(\vec{k})] \quad . \quad . \quad (16)$$

where the N's are diagonal matrices with the characteristic values $0, 1, 2, \ldots$. As was to be expected, H is not positive definite in this case either. We must therefore introduce the LORENTZ condition here too. We thus have to find the quantum mechanical analogues of the two methods of the previous section for an accessory condition $X = 0$ in a quantum mechanical problem characterized by a Hamiltonian H.

First method: Although the most straightforward way would be to interprete $X = 0$ as an operator identity and then to proceed in a similar way as in the corresponding classical treatment, this method is readily seen not to be applicable to the problem on hand, as (13), interpreted in this way, is incompatible with the commutation relations. Thus we shall have to find another treatment. FERMI's idea now is to solve the unrestricted problem

$$H\psi = \frac{i}{\hbar}\,\dot{\psi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and then to look for functionals ψ for which $X\psi = 0$. In general, however, such ψ's do not exist.

Second method: this simply consists in starting from the Lagrangian (7) and thus from the Hamiltonian (14) instead of using (8) and (12) *).

We shall now discuss the applicability of both these methods, first in the MAXWELL, then in the PROCA case.

$\varkappa = 0$. It is well known that here the first method can be used [3]). There appear to exist occupation number functionals ψ, solutions of (17), where H is taken from (12), which vanish when operated on by $B(\vec{k}) - A(0, \vec{k})$ or $B^{\dagger}(\vec{k}) - A^{\dagger}(0, \vec{k})$. For these functionals we will have $N_A(0, \vec{k})\Psi = N_B(\vec{k})\Psi$. The energy (16) reduces therefore to the positive definite form:

$$H = \sum_{\vec{k}} h\,\nu_k\,[N_A(1, \vec{k}) + N_A(2, \vec{k})]. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

*) A less direct way of formulating this method is: start from the *classical* Hamiltonian (12), eliminate with the help of the *classical* condition $X = 0$, (i.e. (13)) one of the variables and then solve the ensuing restricted quantum mechanical problem.

Let us now consider the second method. Thus we have to start from (14), (with $\varkappa = 0$ of course). Quantization of this reduced problem leads at once to (18). This treatment consists essentially in gauging the fields in such a way that the longitudinal and scalar part of the four-vector potential are eliminated before performing the quantization.

$\varkappa \neq 0$. Using the first method, and interpreting (13) accordingly, we would get

$$\bar{\nu}_k^2 \, N_B(\vec{k}) \, \Psi = \nu_k^2 \, N_A(0, \vec{k}) \, \Psi,$$

which is nonsense, as the N's can only be whole numbers. This means that, in the FERMI-interpretation, the two relations (13) are *mutually* incompatible.

Thus the first method has to be discarded (although it can be seen to give a positive definite H), and we must resort to the second one, i.e. we must start with (14). The canonical conjugates of $A(1, \vec{k})$ etc. are $\dot{A}^\dagger(1, \vec{k})$, that of $A(0, \vec{k})$ is $\frac{\varkappa^2}{\bar{\nu}_k^2} \cdot \dot{A}^\dagger(0, \vec{k})$. The commutation relations therefore can be written as:

$$[\dot{A}^\dagger(j, \vec{k}), A(j', \vec{k}')] = -i\hbar \, \delta_{jj'} \, \delta_{\vec{k}\vec{k}'}, \quad j = 1, 2 \quad . \quad . \quad (19a)$$

$$[\dot{A}^\dagger(0, \vec{k}), A(0, \vec{k}')] = -i\hbar \, \nu \, \delta_{\vec{k}\vec{k}'} \cdot \frac{\bar{\nu}_k^2}{\varkappa^2} \cdot \quad . \quad . \quad . \quad . \quad (19b)$$

This yields:

$$H = \sum_{\vec{k}} h \bar{\nu}_k \, [N_A(1, \vec{k}) + N_A(2, \vec{k}) + N_A(0, \vec{k})],$$

which is the desired result.

The relations (19) can be put in a more symmetrical form by performing the following transformation to the new variables $C(\vec{k})$ and $A(3, \vec{k})$:

$$\left.\begin{aligned} \varkappa \, A(0, \vec{k}) &= \bar{\nu}_k \, C(\vec{k}) + \nu_k \, A(3, \vec{k}) \\ \varkappa \, B(\vec{k}) &= \nu_k \, C(\vec{k}) + \bar{\nu}_k \, A(3, \vec{k}). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$A(3, \vec{k})$ and $C(\vec{k})$ are the amplitudes of the plane waves representing free particles with spin 1, the amplitude being directed along the direction of momentum, and spin 0 respectively [5]. The LORENTZ condition in the new variables is simply:

$$C(\vec{k}) = 0, \quad C^\dagger(\vec{k}) = 0.$$

The Hamiltonian becomes:

$$H = \sum_{j=1}^{j=3}{}' \sum_{\vec{k}} [\dot{A}^\dagger(j, \vec{k}) \, \dot{A}(j, \vec{k}) + \bar{\nu}_k^2 \, A^\dagger(j, \vec{k}) \, A(j, \vec{k})].$$

The commutation relations are now:

$$[\dot{A}^{\dagger}(j, \vec{k}), A(j', \vec{k'})] = -i\hbar\,\delta_{jj'}\,\delta_{kk'}, \quad j = 1, 2, 3.$$

and the diagonalized Hamiltonian is:

$$H = \sum_{\vec{k}} h\bar{\nu}_k\,[N_A(1, \vec{k}) + N_A(2, \vec{k}) + N_A(3, \vec{k})].$$

Though we apparently can treat the quantization of the MAXWELL and PROCA case in the same manner, viz. by using the second method, yet the former cannot be derived from the latter by letting $\varkappa$ tend to zero. The reason for this is that the relation (19*b*) would become meaningless or, in other words, that the transformation (20), which is essentially a LORENTZ transformation with $\beta = \nu_k/\bar{\nu}_k$, would become singular.

Comparing, finally, the first method with regard to its applicability to the classical PROCA field on the one, to the same quantized field on the other hand, we remark: the condition $C(\vec{k}) = C^{\dagger}(\vec{k}) = 0$ classically means that we have to put equal to zero an infinite number of oscillator amplitudes, viz. those referring to spinless mesons, for all $\vec{k}$. This, of course, is practicable. But such a procedure is impossible in the quantum mechanical case on account of the zero point vibrations; this is another way of expressing why the first method applies the classical, but not to the quantum PROCA case.

In the MAXWELL case, the transformation (20) loses its sense, as explained above. The LORENTZ condition now amounts to the equality of *two* oscillator amplitudes, (for given $\vec{k}$; cf. (13) with $\bar{\nu}_k = \nu_k$), and this is indeed, also quantum mechanically, possible. Thus we see from (20) how it can be that the first method applies to the quantum mechanical MAXWELL but not to the quantum mechanical PROCA case.

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Mathematics. — *Generalisatie van enkele elementaire stellingen in de n-aire Ω-meetkunde*. By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of October 27, 1945.)

§ 1. *Fundamenteele invarianten.*

In het n-air gebied heeft het puntstelsel $(u' x_i)$ $i = 1, 2, \ldots$ aangevuld met het quadriek $(\Omega' x)^2$ de invarianten

$$(\Omega' \, \Omega)^2 = (\Omega'_1 \, \Omega'_2 \ldots \Omega'_n)^2 = D, \qquad (x_{i_1} \, x_{i_2} \ldots x_{i_n})$$

$$\Omega_{ik} = (\Omega' \, x_i)(\Omega' \, x_k) \qquad i, k = 1, 2, \ldots$$

Uit deze invarianten kan men symmetrische invarianten vormen

$$\triangle (x_{i_1}, x_{i_2}, \ldots x_{i_d}) = \det | \Omega_{rs} | \qquad r, s = i_1, i_2, \ldots i_d.$$

Voor $d > n$ is $\triangle \equiv 0$ wegens de lineaire afhankelijkheid van

$$x_{i_{n+1}}, \ldots x_{i_d} \text{ van } x_{i_1}, x_{i_2}, \ldots x_{i_n}.$$

Voor $d = n$ is

$$\triangle = (\Omega'_1 \, x_{i_1}) \ldots (\Omega'_n \, x_{i_n}) \, . \det | (\Omega'_r \, x_s) | \qquad r, s = i_1, i_2, \ldots i_d$$

$$= \frac{1}{n!} (\Omega'_1 \, \Omega'_2 \ldots \Omega'_n)^2 (x_{i_1} \ldots x_{i_n})^2$$

en derhalve is voor $D \neq 0$ dan en slechts dan $\triangle = 0$ als $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ in één G_m met $m < n$ liggen.

Voor $d < n$ is

$$\triangle = (\Omega'_1 \, x_{i_1}) \ldots (\Omega'_d \, x_{i_d}) \, . \det | (\Omega'_r \, x_s) | \qquad r, s = i_1, i_2, \ldots i_d$$

$$= \frac{1}{d!} (\Omega'_1 \, \Omega'_2 \ldots \Omega'_d \, p'^{\,n-d}_1)(\Omega'_1 \, \Omega'_2 \ldots \Omega'_d \, p'^{\,n-d}_2)$$

als $p^d = \{x_{i_1}, x_{i_2}, \ldots x_{i_d}\} \backsim p'^{\,n-d}$ is. Derhalve is $\triangle = 0$ als

òf de verbindings-G_d van $x_{i_1}, \ldots x_{i_d}$ raakt aan $(\Omega' \, x)^2$

òf $p^d \equiv 0$, dus $x_{i_1}, \ldots x_{i_d}$ in een G_m met $m < d$ liggen.

$\triangle$ is de in n-aire coordinaten overgevoerde invariant voor de deelruimte G_d.
Men kan nu eenvoudige absolute invarianten vormen:

$$\frac{1}{(d-1)!^2} \cdot \frac{\triangle (x_{i_1}, \ldots x_{i_d})}{\Omega_{i_1 i_1} \ldots \Omega_{i_d i_d}} = \sin^2 k_{d-1} \, I (x_{i_1}, \ldots x_{i_d}).$$

waarin k_i constanten zijn en $I(x_{i_1}, \ldots x_{i_d})$ een in $x_{i_1}, \ldots x_{i_d}$ alterneerende grootheid, het $(d-1)$-dimensionale Ω-volume van het simplex, door $x_{i_1}, \ldots x_{i_d}$ gevormd, genoemd worde.

Opmerkingen: 1. Voor $d = 2$ ontstaat

$$\sin^2 k_1 I(x_1, x_2) = \frac{\begin{vmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{vmatrix}}{\Omega_{11}\, \Omega_{22}} = 1 - \frac{\Omega_{12}^2}{\Omega_{11}\, \Omega_{22}}$$

waaruit men ziet, dat $I(x_1, x_2)$ op een constante na de ten opzichte van Ω projectief gemeten afstand van x_1 en x_2 is.

2. Voor $d = n$ ontstaat

$$\sin^2 k_{n-1} I(x_1, \ldots x_n) = \frac{D(x_1\, x_2 \ldots x_n)^2}{(n-1)!^2\, n!\, \Omega_{11} \ldots \Omega_{nn}}.$$

3. Voor kleine waarden van k_1 verkrijgt men met $I(x_i, x_k) = d_{ik}$

$$\sin^2 k_{n-1} I(x_1, \ldots x_n) = \frac{1}{(n-1)!^2} \det|\cos(k_1 d_{ik})| =$$

$$= \begin{vmatrix} 1 & 0 & 0 & . & . \\ 1 & & & & \\ 1 & & 1 - \frac{k_1^2 d_{ik}^2}{2} & & \\ . & & & & \\ . & & & & \end{vmatrix} = \frac{(-1)^n}{(n-1)!^2} \left(\frac{k_1}{2}\right)^{n-1} \begin{vmatrix} \left(\frac{k_1}{2}\right)^2 & 1 & 1 & . & . \\ 1 & & & & \\ 1 & & d_{ik}^2 & & \\ . & & & & \\ . & & & & \end{vmatrix} \rightarrow$$

$$\rightarrow \frac{(-1)^n}{(n-1)!^2} \left(\frac{k_1}{2}\right)^{n-1} \begin{vmatrix} 0 & 1 & 1 & . & . \\ 1 & & & & \\ 1 & & d_{ik}^2 & & \\ . & & & & \\ . & & & & \end{vmatrix}$$

waaruit men ziet, dat voor $k_{n-1} = k_1^{n-1}$ de bekende formule voor volumina in de Euclidische meetkunde ontstaan.

§ 2. *Afstanden van kruisende ruimten.*

Zijn gegeven de kruisende ruimten a^d, b^{n-d} (dus $(a^d b^{n-d}) \neq 0$) en zijn de poolruimten hiervan ten opzichte van Ω respectievelijk $\bar{a}^{n-d}, \bar{b}^d$. De verbindings G_{2d} van a^d en $\bar{b}^d$ snijdt $\bar{a}^{n-d}$ en b^{n-d} elk in een G_d. De vier G_d in deze G_{2d} hebben in het algemeen geval d verschillende (elkaar kruisende) transversalen, gaande door de dekpunten der collineatie ontstaan door projectie uit twee der G_d van de derde G_d op de vierde G_d. Deze transversalen zijn gemeenschappelijke Ω-loodrechten.

Immers — den hoek tusschen twee elkaar snijdende rechten definieerende met behulp van den Ω-afstand van de met het hoekpunt poolverwante punten op de beenen van den hoek — elke rechte door het snijpunt S van één der transversalen met a^d (of b^{n-d}) in deze ruimte getrokken staat

Ω-loodrecht op de transversaal, daar het poolpunt van S op de transversaal in $\bar{a}^{n-d}$ (of $\bar{b}^d$) valt.

Duaal hiermede bestaan $d\,G_{n-2}$ welke gaan door de snijruimte G_{n-2d} van $\bar{a}^{n-d}$ en b^{n-d} en die met elk der ruimten a^d, b^{n-d}, $\bar{a}^{n-d}$, $\bar{b}^d$ in één G_{n-1} liggen. Deze ruimten vindt men in het algemeen geval door de snij-G_{n-2d} te verbinden met telkens $(d-1)$ der transversalen. De transversalen gaan door een poolcorrelatie ten opzichte van Ω over in de laatstgenoemde G_{n-2}.

In het algemeen geval komt onder de transversalen geen Ω-beschrijvende voor en is elke transversaal Ω-poolverwant met een G_{n-2} gaande door de overige. Immers met de transversaal t_i correspondeert een G_{n-2} gaande door de snij-G_{n-2d} en $(d-1)$ transversalen $t_{k_1}, \dots t_{k_{d-1}}$. Komt onder deze t_k nimmer t_i voor, dan is het gestelde juist. Komt t_i in tenminste één geval onder de t_k voor, dan is t_i een Ω-beschrijvende en in $t_{k_1}, \dots t_{k_{d-1}}$ komt tenminste één transversaal, zeg t_p, *niet* voor. De poolruimte van t_p gaat dan niet door t_i en derhalve door t_p. Is dus één der transversalen, t_i, Ω-beschrijvende dan is er nog een andere transversaal, t_p, Ω-beschrijvende. Stel de snijpunten met a^d, b^{n-d}, $\bar{a}^{n-d}$, $\bar{b}^d$ op t_i zijn A, B, C, D, die op t_p A', B', C', D'. De poolruimten van A, B, C, D snijden dan t_p in C', D', A', B' en er geldt voor de dubbelverhoudingen

$$DV(A\,B\,C\,D) = DV(C'\,D'\,A'\,B') = DV(A'\,B'\,C'\,D')$$

en de rechten AA', BB', CC', DD' liggen hyperboloïdisch. Er zijn ∞-veel (Clifford-parallelle) transversalen, in tegenspraak met het ondersteldE.

In het algemeen geval is het derhalve mogelijk in a^d de punten $x_1, x_2, \dots x_d$ te kiezen en in b^{n-d} de punten $\xi_1, \xi_2, \dots \xi_d$, $\alpha_1, \alpha_2, \dots \alpha_{n-2d}$ zóó, dat daaronder geen punten van Ω voorkomen en alle puntkoppels behoudens (x_i, ξ_i) $i-1, 2, \dots d$ poolkoppels ten opzichte van Ω zijn: men kieze slechts x_i, ξ_i als snijpuntenpaar met de transversaal t_i en voor $\alpha_1, \dots \alpha_{n-2d}$ een poolsimplex in de snij-G_{n-2d} van $\bar{a}^{n-d}$ en b^{n-d}.

Nu wordt $\dfrac{\det|\Omega_{pq}|}{\Omega_{pp}\dots\Omega_{qq}}$ $p, q = (x_1, \dots \xi_1, \dots \alpha_1, \dots)$ eenerzijds gelijk aan $\prod\limits_{i=1}^{d} \sin^2 k_1 l(x_i, \xi_i)$ en anderzijds is

$$\det|\Omega_{pq}| = \frac{D(x_1 \dots \xi_1 \dots \alpha_1 \dots)^2}{n!} = \frac{D(a^d\, b^{n-d})(a_1^d\, b_1^{n-d})}{n!}$$

$$\Omega_{x_1x_1} \dots \Omega_{x_dx_d} = \det|\Omega_{x_ix_k}| = \frac{1}{d!}(\Omega_1'\,\Omega_2' \dots \Omega_d'\, a'^{\,n-d})(\Omega_1'\,\Omega_2' \dots \Omega_d'\, a_1'^{\,n-d})$$

en derhalve is het product der quadraten der sinus der d afstanden, δ_i, van a^d en b^{n-d} gegeven door

$$\prod_{i=1}^{d} \sin^2 k_1\, \delta_i = \frac{d!\,(n-d)!}{n!} \cdot \frac{D(a^d\, b^{n-d})(a_1^d\, b_1^{n-d})}{(\Omega_1' \dots \Omega_d'\, a'^{\,n-d}) \dots (\Omega_{d+1}' \dots b_1'^{\,d})}.$$

Voor $d = 1$ voert deze formule voor den afstand δ van een punt x tot een G_{n-1} v' naar

$$\sin^2 k_1 \delta = \frac{D(xv')^2}{n(\Omega' x)^2 (\Omega v')^2}.$$

§ 3. *Simplexvolumina.*

Voor een willekeurig simplex geldt nu wanneer men de hoekpunten in twee groepen verdeelt

$$\underbrace{x_1, x_2 \ldots x_d} \quad ; \quad \underbrace{x_{d+1}, \ldots x_n} \qquad d > 1$$

$$\sin^2 V = \sin^2 k_{n-1} I(x_1, \ldots x_n) = \frac{\triangle(x_1, \ldots x_n)}{(n-1)!^2 \, \Omega_{11} \ldots \Omega_{nn}} = \frac{D(x_1, x_2 \ldots x_n)^2}{(n-1)! \, n! \, \Omega_{11} \ldots \Omega_{nn}}$$

$$\sin^2 B = \sin^2 k_{d-1} I(x_1, \ldots x_d) = \frac{\triangle(x_1, \ldots x_d)}{(d-1)!^2 \, \Omega_{11} \ldots \Omega_{dd}} =$$

$$= \frac{(\Omega'_1 \ldots \Omega'_d \, p'^{\,n-d})(\Omega'_1 \ldots \Omega'_d \, p_1'^{\,n-d})}{(d-1)!^2 \, d! \, \Omega_{11} \ldots \Omega_{dd}}, \qquad \{x_1, \ldots x_d\} = p^d$$

$$\sin^2 G = \sin^2 k_{n-d-1} I(x_{d+1}, \ldots x_n) =$$

$$= \frac{(\Omega'_{d+1} \ldots \Omega'_n \, q'^{\,d})(\Omega'_{d+1} \ldots \Omega'_n \, q_1'^{\,n-d})}{(n-d-1)!^2 \, (n-d)! \, \Omega_{d+1,d+1} \ldots \Omega_{nn}}, \qquad \{x_{d+1}, \ldots x_n\} = q^{n-d}$$

waaruit

$$\sin^2 V = \frac{(n-d-1)!^2 \, (d-1)!^2}{(n-1)!^2} \sin^2 B \sin^2 G \prod_{i=1}^{d} \sin^2 k_1 \delta_i.$$

dus

$$\pm \sin V = \frac{(n-d-1)! \, (d-1)!}{(n-1)!} \sin B \sin G \prod_{i=1}^{d} \sin^2 k_1 \delta_i$$

waardoor, afgezien van de maatconstanten k_i de sinus van het volume gegeven wordt als product van sinus van „grond"- en „boven"-simplex maal het product der sinus der afstanden van „grond"- en „boven"-ruimte.

Voor $d = 1$ ontstaat

$$\sin^2 V = \frac{D(x_1 \ldots x_n)^2}{(n-1)!^2 \, n! \, \Omega_{11} \ldots \Omega_{nn}} =$$

$$= \frac{1}{(n-1)^2} \cdot \frac{1}{(n-1)!(n-2)!^2} \frac{(\Omega v')^2}{\Omega_{22} \ldots \Omega_{nn}} \cdot \frac{1}{n} \cdot \frac{D(xv')^2}{\Omega_{11} (\Omega v')^2} \qquad v' = \{x_2 \ldots x_n\}$$

òf

$$\pm \sin V = \frac{1}{n-1} \sin G \sin \delta.$$

Opmerking: Voor $n=3$ levert $I(x_1, x_2, x_3)=0$ op grond der relatie

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=$$
$$=(\cos(\alpha-\beta)-\cos\gamma)(\cos\gamma-\cos(\alpha+\beta))$$

het additietheorema van afstanden terug.

§ 4. *Additietheorema van volumina. Generalisatie van het theorema van* STEWART.

In de projectieve ruimte bestaat de fundamenteele identiteit:

$$(u'\,t)(x_1\,x_2\ldots x_n)=(u'\,x_1)(t\,x_2\ldots x_n)+$$
$$+(u'\,x_2)(x_1\,t\,x_3\ldots x_n)+\ldots(u'\,x_n)(x_1\,x_2\ldots x_{n-1}\,t) \qquad \{u'\}.$$

Hieruit volgt voor $u'=$ poolruimte van een punt z

$$\Omega_{zt}(x_1\,x_2\ldots x_n)=\Omega_{zx_1}(t\,x_2\ldots x_n)+\ldots\Omega_{zx_n}(x_1,\ldots x_{n-1}\,t)$$

of na multiplicatie met een quadraatwortel uit

$$\frac{D}{n!\,(n-1)!^2\,\Omega_{11}\ldots\Omega_{nn}\,\Omega_{zz}\,\Omega_{tt}}$$

de fundamenteele relatie voor de Ω-meetkunde:

$$\cos k_1 I(z,t)\sin k_{n-1} I(x_1\ldots x_n)=\sum_{i=1}^{n}\cos k_1 I(z,x_i)\sin k_{n-1} I(x_1,\ldots t,\ldots x_n).$$

a. Stelt men hierin $z\equiv t$ dan ontstaat het *additietheorema van volumina*

$$\sin k_{n-1} I(x_1\ldots x_n)=\sum_{i=1}^{n}\cos k_1 I(z,x_i)\sin k_{n-1} I(x_1,\ldots z,\ldots x_n).$$

b. Ligt t in $\{x_1,\ldots x_{d-1}\}=v'$ en is $z\equiv x_n$ dan ontstaat

$$\cos k_1 I(x_n,t)\sin k_{n-1} I(x_1,\ldots x_n)=\sum_{i=1}^{n-1}\cos k_1 I(x_n,x_i)\sin k_{n-1} I(x_1,\ldots t,\ldots x_n)$$

want $\sin k_{d-1} I(x_1, x_2, \ldots x_{n-1}, t)=0$, hetgeen op grond van het feit, dat alle simplices het hoekpunt x_n bezitten, na deeling door een getallenfactor maal den afstand van x_n tot v' leidt tot:

$$\cos k_1 I(x_n,t)\sin k_{n-2} I(x_1,\ldots x_{n-1})=\sum_{i=1}^{n-1}\cos k_1 I(x_n,x_i)\sin k_{n-2} I(x_1,\ldots t,\ldots x_{n-1})$$

de generalisatie van het theorema van STEWART.

Volgens het additietheorema van de „basis”-ruimte v' kan dit omgevormd worden tot

$$\{1-\cos k_1 I(x_n,t)\}\sin k_{n-2} I(x_1,\ldots x_{n-1})=$$
$$=\sum_{i=1}^{n-1}\{\cos k_1 I(t,x_i)-\cos k_1 I(x_n,x_i)\}\sin k_{n-2} I(x_1,\ldots t,\ldots x_{n-1})$$

waaruit men voor de Euclidische meetkunde terugvindt

$$\underline{x_n t^2 \,.\, Vol(x_1 \ldots x_{n+1}) = \Sigma(\overline{x_n x_i}^2 - \overline{t x_i}^2) \,.\, Vol(x_1, x_2, \ldots t_1 \ldots x_{n-1}).}$$

Toelichting (fig. 1): Voor $n = 3$ luidt het additietheorema, afziende van maatconstanten,

$$\sin ABC = \cos \xi \sin I + \cos \eta \sin II + \cos \zeta \sin III.$$

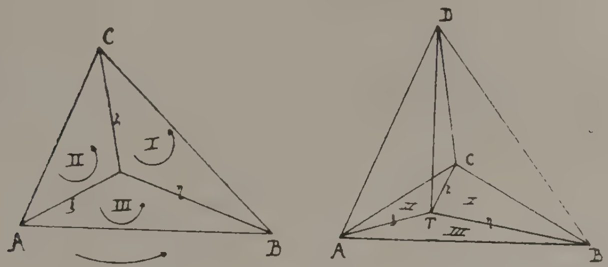


Fig. 1.

Voor $n = 4$ het theorema van STEWART

$$\cos \overline{DT} \sin \overline{ABC} = \cos \overline{AD} \sin I + \cos \overline{BD} \sin II + \cos \overline{CD} \sin III$$

waaruit voor $D \equiv T$ het additietheorema voor $n = 3$ terugverkregen wordt.

§ 5. *Generalisatie van den cosinusregel.*

In de volgende beschouwingen worden de formules eenvoudigheidshalve uitgeschreven met $k_i \equiv 1 \; \{ i \}$.

a. $n = 3$.

In de Ω-meetkunde van het platte vlak wordt voor een in C Ω-rechthoekigen driehoek ABC door de zijden van den driehoek en de poollijnen van de hoekpunten A en B het vlak verdeeld in één vijfhoek, vijf vierhoeken en vijf driehoeken. De relatie van NEPER voor de Ω-meetkunde leest men hieruit af (fig. 2): In de betrekkingen tusschen zijden en hoeken van een Ω-rechthoekigen driehoek kan men de volgende permutaties uitvoeren van

$$c - \beta - a - b - \alpha \quad (I) \text{ in}$$
$$\beta - \bar{a} - b - \bar{\alpha} - c \quad (II)$$
$$\bar{a} - \bar{b} - \bar{\alpha} - \bar{c} - \beta \quad (III)$$
$$\bar{b} - \alpha - \bar{c} - \bar{\beta} - \bar{a} \quad (IV)$$
$$\alpha - c - \beta - a - b \quad (V),$$

waarbij de complementvorming aangeduid wordt door overstreeping.

De fundamenteele betrekking is, zooals bekend,

$$\pm \cos c = \cos a \cos b,$$

welke door quadrateeren omgevormd kan worden tot

$$\sin^2 c = \sin^2 a + \sin^2 b - \sin^2 a \sin^2 b = \sin^2 a + \sin^2 b - 4 \sin^2 V$$

„Pythagoras"

Voor een willekeurigen driehoek ontstaat

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

waaruit door quadrateeren volgt:

$$(\cos c - \sin a \sin b \cos C)^2 = \cos^2 a \cos^2 b$$
$$\sin^2 c = \sin^2 a + \sin^2 b - 2 \sin a \sin b \cos C \cos c - 4 \sin^2 \mathfrak{V}$$

„Cosinusregel"

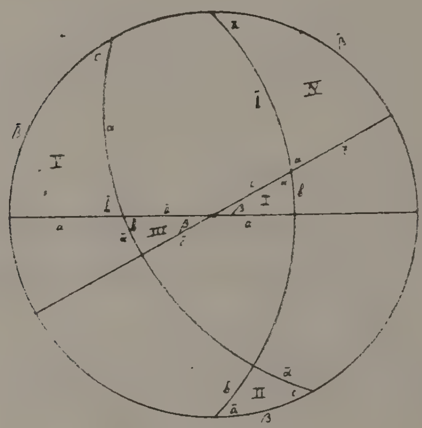


Fig. 2.

Het opnieuw optreden van c in het rechterlid vindt zijn oorzaak in het feit, dat voor $n = 3$ de „ribben" tevens „zij-ruimten" van het simplex zijn.

b. $n > 3$.

Zij a_n de „top" van het simplex, $\mathfrak{A}_i$ de overstaande zijruimte van a_i, A_i de projectie van $\mathfrak{A}_i$ op $\mathfrak{A}_n$, α_i de afstand van a_i tot het voetpunt L der hoogtelijn h uit a_n, p_i de hoogtelijn uit a_n in $\mathfrak{A}_i$, q_i de hoogtelijn in A_i uit L.

Volgens het additietheorema der volumina geldt dan

$$\sin \mathfrak{A}_n = \sum_i \cos \alpha_i \sin A_i = \sum_i \cos \alpha_i \frac{\sin \mathfrak{A}_i \sin q_i}{\sin p_i} =$$

$$= \sum_i \sin \mathfrak{A}_i \frac{\cos (a_i, a_n)}{\cos h} \cdot \frac{\operatorname{tg} q_i}{\operatorname{tg} p_i} \cdot \frac{\cos q_i}{\cos p_i} = \sum_i \frac{\sin \mathfrak{A}_i \cos (a_i, a_n) \cos (\mathfrak{A}_i, \mathfrak{A}_n)}{\cos^2 h}$$

of

$$\underline{\sin^2 \mathfrak{A}_n - (n-1)^2 \sin^2 \mathfrak{V} = \sum_i \sin \mathfrak{A}_i \sin \mathfrak{A}_n \cos (a_i, a_n) \cos (\mathfrak{A}_i, \mathfrak{A}_n).}$$

Sommeert men de hiermede analoge formules voor $\mathfrak{A}_1, \mathfrak{A}_2, \dots \mathfrak{A}_{n-1}$ dan ontstaat

$$\sum_{i=1}^{n-1} \sin^2 \mathfrak{A}_i - (n-1)^3 \mathfrak{B} =$$

$$= 2 \sum_{(i,k)=1}^{n-1} \sin \mathfrak{A}_i \sin \mathfrak{A}_k \cos(a_i, a_k) \cos(\mathfrak{A}_i, \mathfrak{A}_k) + \sin^2 \mathfrak{A}_n - (n-1)^2 \sin^2 \mathfrak{B}$$

of tenslotte

$$\underline{\sin^2 \mathfrak{A}_n = \sum_{i=1}^{n-1} \sin^2 \mathfrak{A}_i - 2 \sum_{(i,k)=1}^{n-1} \sin \mathfrak{A}_i \sin \mathfrak{A}_k \cos(a_i, a_k) \cos(\mathfrak{A}_i, \mathfrak{A}_k) - (n-1)^2 (n-2) \sin^2 \mathfrak{B}.}$$

„Gegeneraliseerde cosinusregel"

Voor het geval, dat de ribben aan één hoekpunt, a_n, alle onderling Ω-orthogonaal zijn volgt

$$\underline{\sin^2 \mathfrak{A}_n = \sum_{i=1}^{n-1} \sin^2 \mathfrak{A}_i - (n-2)(n-1)^2 \sin^2 \mathfrak{B}}$$

„Pythagoras"

waarbij men desgewenscht $\sin \mathfrak{B}$ nog in $\mathfrak{A}_i$ kan uitdrukken volgens

$$(n-2)!^n \prod_{i=1}^{n-1} \sin \mathfrak{A}_i = \{(n-1)! \sin \mathfrak{B}\}^{n-2}.$$

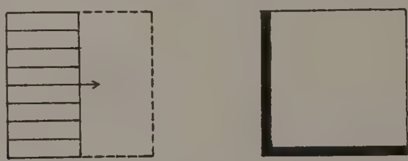
Mathematics. — *Over de benadering van $\frac{\pi}{4}$ in de Aegyptische meetkunde.*

By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of October 27, 1945.)

§ 1. In het onderstaande trachten wij aan te toonen, dat de in de Aegyptische wiskunde optredende benadering $\frac{\pi}{4} = \frac{64}{81}$ langs theoretischen weg door de Aegyptenaren is verkregen. Om tot deze slotsom te kunnen komen, wijzen wij met STRUVE [1]) allereerst op de consequente wijze, waarmede de terminologie werd toegepast en op de wijze, waarop oplosbare vergelijkingen van het type $ax^2 = b$ werden behandeld om hieruit de quadratuur van den cirkel te verkrijgen.

1. De Aegyptenaren kenden het begrip quadrateeren en eveneens den quadraatwortel. Terwijl voor de aanduiding der vermenigvuldiging de term *sp* = maal gebruikt wordt en in het bijzonder voor tweemaal nemen = verdubbelen de term *ḳ3b* in gebruik is, leest men voor quadrateeren: *ỉr-ḫr-k x m sn* = reken j ij met *x* als voorbijloopende = terwijl voorbijgeschoven wordt, hetgeen op een zuiver meetkundigen oorsprong der uitdrukking wijst. Hiermede overeenkomende wordt de quadraatwortel aangeduid met *ḳnb-t*, de hoek, de zijde van het quadraat (zie fig. 1). Een aanwijzing voor de



ỉr-t x m sn *ḳnb-t*

Fig. 1.

consequentie, waarmede de terminologie wordt toegepast vindt men (STRUVE) in de beroemde opgave over de berekening van het volume van een afgeknotte pyramide, waarin men bij de berekening der termen van $a^2 + ab + b^2$ voor $a = 4$ en $b = 2$ leest [in den papyrus van Moskou, kolom XXVII 4, 5, 6]:

4. *ỉr-ḫr-k* 4 *pn m sn ḫpr* 16	Bereken het quadraat van deze 4. Er komt 16.
5. *ỉr-ḫr-k ḳ3b-k* 4 *ḫpr* 8	Bereken en verdubbel 4. Er komt 8.
6. *ỉr-ḫr-k* 2 *pn m sn ḫpr* 4	Bereken het quadraat van deze 2. Er komt 4.

Hieruit ziet men, dat 2 maal 4 als „verdubbelen" wordt omschreven, terwijl 2^2 niet als $2 + 2$ of 2 maal $2 =$ „2 verdubbelen" wordt aangewezen.

[1]) W. STRUVE, Quellen und Studien A 1, pag. 135.

In verband hiermede moet er ons inziens groote nadruk vallen op het feit, dat de berekening van de oppervlakte van den cirkel C niet als een fractie van het omgeschreven quadraat $V : 64/81\ V$, dat is $(2/3 + 1/9 + 1/81)\ V$ doch als oppervlakte van een vierkant met zijde $8/9\ d$, als d den diameter voorstelt, wordt berekend.

2. Hoe de vergelijking $ax^2 = b$ opgelost werd kan onder meer blijken uit kolom VIII, IX, XXXIII—XXXIV uit den papyrus van Moskou, waarvan wij duidelijkheidshalve kolom IX citeeren.

tp n ỉr-t spd-t
mỉ ḏd nk spd-t nt 3ḫt 20 *ỉdb n* 2½.
ỉr-ḫr-k ḳ3b-k 3ḫt ḫpr-ḫr 40 *ỉr sp* 2½.
ḫpr-ḫr 100 *ỉr knb ḫpr-ḫr* 10 *njs wʿ ḫnt* 2½.
ḫpr-r-ḫr ỉm pw 1/3 1/15 *ỉr* 1/3 1/15 *n* 10 *ḫpr-ḫr* 4.
10 *pw n 3w n* 4 *m sḫw*.

Vertaald:

Methode ter berekening van een rechthoekigen driehoek.

Als aan je gezegd wordt een rechthoekigen driehoek met oppervlakte 20 en verhouding der rechthoekszijden van 2½.

Verdubbel j ij de oppervlakte. Er komt 40. Reken 2½ maal.

Er komt 100. Bereken de wortel. Er komt 10. Roep 1 voor 2½ (= deel 1 door 2½).

Er komt daar dit: 2/5. Bereken 2/5 van 10. Er komt 4.

10 is het als lengte voor 4 als breedte.

In moderne symbolen:

Opgave: $x = 2\frac{1}{2}y \quad \frac{1}{2}xy = 40.$

Oplossing: $2 . 20 = 40 = xy \quad 2\frac{1}{2}xy = x^2 = 2\frac{1}{2} . 40 = 100 \quad x = 10.$
$2\frac{1}{2}y = 10 \quad 1 : 2\frac{1}{2} = 2/5 \quad 2/5 . 2\frac{1}{2}y = y = 2/5 . 10 = 4.$

De vergelijking $y = ax = b$ wordt dus opgelost door beide leden der vergelijking te vermenigvuldigen met $1/a$, terwijl als het ware $ay^2 = b$ opgelost wordt door $a . ay^2 = (ay)^2 = ab$ enz.

3. Bedenkt men nu verder, dat de bewerkingen voor de eenvoudige oppervlakteberekeningen vaak worden gemotiveerd met „om er een rechthoek van te maken" dan ligt het voor de hand, dat de berekening van de oppervlakte van den cirkel geleid heeft tot de vraag naar „zijn hoek", naar de zijde van het vierkant met dezelfde oppervlakte.

De figuur uit den papyrus Rhind R. 48 levert aanwijzing voor de identificatie van den cirkel met den achthoek ontstaan door de hoeken van een omgeschreven vierkant op 1/3 af te knotten (zie fig. 2). Hieruit volgt de relatie

$$9\,C = 7\,V$$

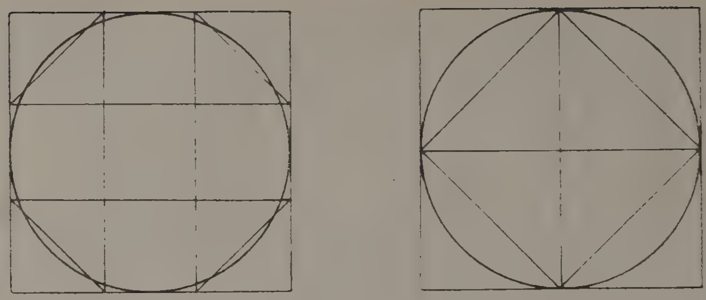


Fig. 2.

onmiddellijk en dus als d de zijde van het quadraat om den cirkel en x de zijde van het quadraat met gelijke oppervlakte is

$$9\,x^2 = 7\,d^2$$

waaruit

$$9\,.\,9x^2 = (9x)^2 = 9\,.\,7\,d^2 = 63\,d^2 \sim (8d)^2$$

dus

$$9x = 8d \qquad x = d - 1/9\,d$$

en

$$\frac{\pi}{4} = \left(\frac{8}{9}\right)^2 = \frac{64}{81}$$

onmiddellijk volgt, terwijl

$$C = \left[\left(d - \frac{1}{9}d\right) - \frac{1}{9}\left(d - \frac{1}{9}d\right)\right]d.$$

§ 2. Nader onderzoek der methode van moderner standpunt.

1. Verdeelt men het omgeschreven quadraat in vier deelvierkanten door halveering der zijden, dan verkrijgt men (zie fig. 2) onmiddellijk

$$4 > C > 2$$

in deelvierkanten uitgedrukt, dus $\pi \sim 3$; $\frac{\pi}{4} = 0{,}75$.

Nú levert de „quadratuur” van den cirkel

$$4\,x^2 = 3\,d^2$$

waaruit allereerst de, te verwerpen, oplossing $x = d = 1$ volgt en verder

$$\begin{array}{rl} 4 \text{ maal:} & 16\,x^2 = 12\,d^2 \\ 9 \text{ maal:} & 36\,x^2 = 27\,d^2 \\ 16 \text{ maal:} & 64\,x^2 = 48\,d^2 \sim (7d)^2 \qquad 8\,x = 7\,d \end{array}$$

$$\frac{\pi}{4} = \frac{49}{64} = 0{,}765.$$

2. De verdeeling in drie gelijke deelen der zijden van het omgeschreven vierkant levert, zooals boven reeds werd aangegeven

$$9\,x^2 = 7d^2$$

waarvan de „eerste oplossing" is $x = 9$, $d = 8$ met $\frac{\pi}{4} = \frac{64}{81} = 0{,}790$, $\pi = 3{,}16$ tegenover $\pi/4 = 7/9$, $\pi = 3{,}11$ uit de directe vergelijking der oppervlakten.

De kettingbreuk $\sqrt{\frac{\pi}{4}} = \{0;\ 1,\ 7,\ 1,\ 3,\ 1,\ 2,\ 1,\ \ldots\}$

met de naderende breuken

$$\frac{1}{1};\ \frac{7}{8};\ \frac{8}{9};\ \frac{31}{35};\ \ldots$$

doet zien, dat de methode bij halveering der zijde de eerste twee naderende breuken oplevert, terwijl de deeling in drieën de derde breuk doet vinden. Voor de oppervlakte als deel van het omgeschreven vierkant volgt evenzoo

$$\frac{\pi}{4} = \{0;\ 1,\ 3,\ 1,\ 1,\ \ldots\}$$

$$\frac{1}{1};\ \frac{3}{4};\ \frac{4}{5};\ \frac{7}{9};\ \ldots$$

waarvan dus de tweede en vierde breuk uit de figuur onmiddellijk afgelezen worden.

Opmerking: De hierboven aangegeven afleiding van de berekening van de oppervlakte van den cirkel, doet vermoeden, dat de formule voor het volume van een pyramide eveneens door „berekening" zal zijn verkregen. Door de eenvoudige breuk 1/3, die door schatting eveneens te verklaren is, kan hiervoor niet een dergelijke aanwijzing worden verkregen als voor de oppervlakte van den cirkel. Echter, gebruik makende van het gegeven, dat het begrip „fractie" en (zelfs) het begrip „verhouding" in de papyri gevonden worden, kan men, uitgaande van de formule voor de afgeknotte pyramide

$$V = 1/3\,h\,(a^2 + ab + b^2)$$

voor een afgeknotte pyramide met quadratisch grondvlak met zijde a en een boven-vlak met zijde b, die, zooals NEUGEBAUER [2]) aangegeven heeft, door verdeeling uit de formule voor den inhoud van een pyramide met quadratisch grondvlak gemakkelijk te verkrijgen is, óók deze formule zelf verkrijgen.

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[2]) O. NEUGEBAUER, Vorlesungen über vorgr. Math. etc., pag. 128.

Allereerst ziet men in figuur 3 de verdeeling aangegeven door NEUGEBAUER voor het geval, dat één der opstaande ribben vertikaal staat:

$$V = b^2h + 2 \,.\, \tfrac{1}{2}\, b\,(a-b)\,h + \tfrac{1}{3}\, h\,(a-b)^2 = \tfrac{1}{3}\, h\,(a^2 + ab + b^2),$$

terwijl wij voor de afgeknotte regelmatige pyramide tot dezelfde algebraïsche uitdrukking komen

$$V = b^2h + 4 \,.\, \tfrac{1}{2} \,.\, \tfrac{1}{2}\,(a-b)\,bh + 4 \,.\, \tfrac{1}{4}\,(a-b)^2 \,.\, \tfrac{1}{3}\, h = \tfrac{1}{3}\, h\,(a^2 + ab + b^2)\,.$$

Nu vindt men evenzoo, uit de splitsing voor de pyramide met vertikale opstaande ribbe, door deze te beschouwen als *fractie van het blok* met

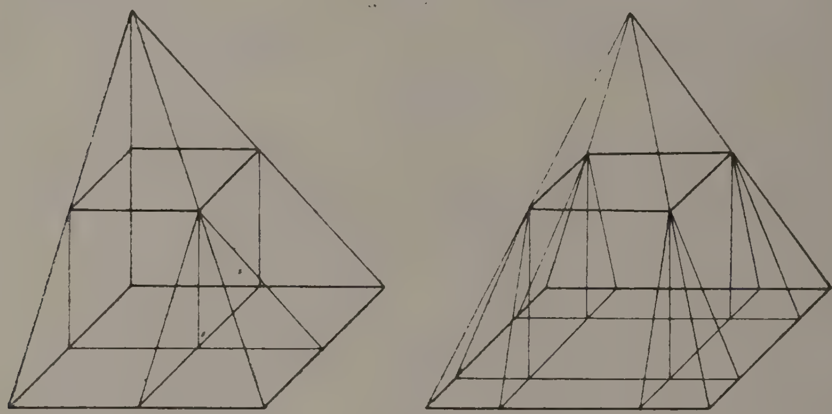

Fig. 3.

dezelfde ribben aan den orthogonalen driebeen èn af te knotten op de halve hoogte; als J_1 de fractie x van het Blok B, met ribben a, b, c door de pyramide geleverd voorstelt en J_2, de pyramide met ribben $2a$, $2b$, $2c$, dezelfde fractie van het blok $B_2 = 8\,B_1$ is

$$J_2 = 2\,J_1 + 2\,B_1 = xB_2.$$

Dus voor x de vergelijking

$$2 + 2\,x = 8\,x. \qquad x = 1/3.$$

Mathematics. — *On the absolute convergence of Fourier series.* By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 27, 1945.)

S. BERNSTEIN [1]) has proved that if the real, periodic function $f(x)$ (period 2π) satisfies a Lipschitz-condition of order α, where $\alpha > \frac{1}{2}$, then the Fourier series of $f(x)$ converges absolutely. For $\alpha = \frac{1}{2}$ this is no longer true. O. SZÁSZ [2]) generalized this theorem by considering the series

$$\sum_{n=1}^{\infty} (|a_n|^{\beta} + |b_n|^{\beta}), \quad \ldots \ldots \ldots \quad (1)$$

where a_n and b_n are the Fourier coefficients of $f(x)$, and showing that if $f(x)$ satisfies a Lipschitz-condition of order $\alpha (0 < \alpha \leq 1)$, the series (1) converges for every $\beta > 2/(2\alpha + 1)$, but not necessarily for $\beta = 2/(2\alpha + 1)$. Recently L. NEDER [3]) made the following addition to BERNSTEIN's Theorem: If, for $h > 0$,

$$l_1(h) = \log(e + h^{-1}),$$
$$l_2(h) = \log\log(e^e + h^{-1}),$$
$$\text{etc.},$$

and if, for a certain $\varepsilon > 0$,

$$|f(x+h) - f(x)| \leq \frac{C\, h^{1/2}}{l_1(h)\, l_2(h) \ldots l_k(h)^{1+\varepsilon}},$$

then the Fourier series of $f(x)$ converges absolutely.

We shall prove a similar addition to SZÁSZ's Theorem.

Theorem 1. *If* $0 < \alpha \leq 1$, $\varepsilon > 0$, $h > 0$, *and*

$$|f(x+h) - f(x)| \leq \frac{C\, h^{\alpha}}{[l_1(h)\, l_2(h) \ldots l_k(h)^{1+\varepsilon}]^{\frac{2\alpha+1}{2}}}, \quad \ldots \quad (2)$$

then the series (1) *converges also for* $\beta = 2/(2\alpha + 1)$.

Proof. We shall give the proof for $k = 2$. If

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

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[1]) S. BERNSTEIN, Sur la convergence absolue des séries trigonométriques, C. R. de l'Acad de Sc. de Paris 158 (1914), 1661—1664.

[2]) O. SZÁSZ, Über den Konvergenzexponent der Fourierschen Reihen. Münchener Sitzungsberichte (1922), 135—150.

[3]) L. NEDER, Ein Satz über die absolute Konvergenz der Fourier-Reihe, Math. Zeitschrift 49 (1944), 644—646.

then

$$f(x+h)-f(x-h) \sim 2 \sum_{n=1}^{\infty} b_n(x) \sin nh,$$

where

$$b_n(x) = b_n \cos nx - a_n \sin nx,$$

so that

$$\frac{1}{\pi} \int_0^{2\pi} [f(x+h)-f(x-h)]^2 \, dx = 4 \sum_{n=1}^{\infty} \varrho_n^2 \sin^2 nh,$$

where

$$\varrho_n^2 = a_n^2 + b_n^2.$$

From (2) it follows now that

$$\frac{1}{\pi} \int_0^{2\pi} [f(x+h)-f(x-h)]^2 \, dx \leqslant \frac{C_1 h^{2\alpha}}{[l_1(2h)\, l_2(2h)^{1+\varepsilon}]^{2\alpha+1}},$$

so that, taking $h = \pi/2N$,

$$\sum_{n=1}^{N} \varrho_n^2 \sin^2 \frac{\pi n}{2N} \leqslant \frac{C_2 N^{-2\alpha}}{\left[l_1\left(\frac{\pi}{N}\right) l_2\left(\frac{\pi}{N}\right)^{1+\varepsilon} \right]^{2\alpha+1}}.$$

Let now $N = 2^\nu$, where ν is an integer, $\geqslant \nu_0 \geqslant (\log 2)^{-2} + 3$. Then

$$l_1\left(\frac{\pi}{N}\right) = \log\left(e + \frac{2^\nu}{\pi}\right) > \log 2^{\nu-2} = (\nu-2)\log 2,$$

$$l_2\left(\frac{\pi}{N}\right) = \log\log\left(e^e + \frac{2^\nu}{\pi}\right) > \log\log 2^{\nu-2} = \log(\nu-2) + \log\log 2 > \tfrac{1}{2}\log(\nu-2),$$

hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^2 \leqslant \frac{C_3 2^{-2\nu\alpha}}{[(\nu-2)\log^{1+\varepsilon}(\nu-2)]^{2\alpha+1}},$$

and from this, by HÖLDER's inequality

$$\Sigma \varrho_n^\beta \leqslant (\Sigma \varrho_n^2)^{\beta/2} (\Sigma 1)^{1-\beta/2}$$

with $\beta = 2/(2\alpha+1)$, so that $1 - \beta/2 = 2\alpha/(2\alpha+1)$,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^\beta \leqslant \frac{C_4 2^{-\frac{2\nu\alpha}{2\alpha+1}}}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} \cdot 2^{\frac{2\nu\alpha}{2\alpha+1}} = \frac{C_4}{(\nu-2)\log^{1+\varepsilon}(\nu-2)}.$$

This shows finally that

$$\sum_{n=2^{\nu_0-1}+1}^{\infty} \varrho_n^{\beta} = \sum_{\nu=\nu_0}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \varrho_n^{\beta} \leqslant C_4 \sum_{\nu=\nu_0}^{\infty} \frac{1}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} < \infty.$$

The series $\sum \varrho_n^{\beta}$ converges therefore, and, since $|a_n|^{\beta}$, and also $|b_n|^{\beta}$, does not exceed $\varrho_n{}^{\beta}$, the same is true of the series

$$\sum_{n=1}^{\infty}(|a_n|^{\beta} + |b_n|^{\beta}).$$

It may be asked whether there exist functions $f(x)$, satisfying a condition of the form (2), without belonging to a Lipschitz-class of order $\alpha' > \alpha$. We shall show this to be the case.

Theorem 2. *If*

$$0 < \alpha < 1, \quad \varepsilon > 0, \quad \gamma = \frac{2\alpha + 1}{2}, \quad \delta = 1 + \gamma(1 + \varepsilon),$$

the function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{i n \log n}}{n^{\gamma}(\log n)^{\delta}} e^{i n x}$$

satisfies the condition

$$|f(x+h) - f(x)| \leqslant \frac{C h^{\alpha}}{[l_1(h)^{1+\varepsilon}]^{\gamma}},$$

while, for $\alpha' > \alpha$, $f(x)$ does not belong to the Lipschitz-class of order α'. The real and imaginary components of $f(x)$ may therefore serve as illustrations to Theorem 1.

Proof. Writing

$$s_{\nu}(x) = \sum_{n=1}^{\nu} e^{i n \log n} e^{i n x},$$

it may be proved [4] that

$$s_{\nu}(x) = O(\nu^{1/2}) \text{ and } s_{\nu}'(x) = \frac{d s_{\nu}(x)}{dx} = O(\nu^{3/2}),$$

uniformly in x. Furthermore

$$\int_N^{\infty} \frac{dx}{x^{\alpha+1}\log^{\delta} x} \leqslant \frac{1}{N^{\alpha}} \int_N^{\infty} \frac{dx}{x \log^{\delta} x} = \frac{1}{N^{\alpha}} \int_{\log N}^{\infty} \frac{dy}{y^{\delta}} = O\left(\frac{N^{-\alpha}}{\log^{\delta-1} N}\right). \quad (3)$$

and

$$\int_2^N \frac{dx}{x^{\alpha}\log^{\delta} x} = \int_2^N \frac{x^{1-\alpha}\,dx}{x \log^{\delta} x} < N^{1-\alpha} \int_{\log 2}^{\log N} \frac{dy}{y^{\delta}} = O\left(\frac{N^{1-\alpha}}{\log^{\delta-1} N}\right). \quad (4)$$

[4]) See e.g. A. ZYGMUND, Trigonometrical Series, Warsaw (1935), 5. 32.

Introducing the difference $\Delta g(\nu) = g(\nu) - g(\nu+1)$, we find now by ABEL's transformation

$$f(x+h)-f(x)=\sum_{\nu=1}^{\infty}\{s_\nu(x+h)-s_\nu(x)\}\,\Delta\nu^{-\gamma}\log^{-\delta}\nu=\sum_{\nu=1}^{N}+\sum_{\nu=N+1}^{\infty}=P+Q,$$

where $N=\left[\frac{1}{h}\right]$. Since, by the mean-value theorem,

$$\Delta\,\nu^{-\gamma}\log^{-\delta}\nu = O(\nu^{-\gamma-1}\log^{-\delta}\nu), \text{ and } s_\nu(x)=O(\nu^{1/2}),$$

the terms in Q are

$$O(\nu^{1/2-\gamma-1}\log^{-\delta}\nu)=O(\nu^{-\alpha-1}\log^{-\delta}\nu),$$

so that, on account of (3),

$$|Q|=\Big|\sum_{N+1}^{\infty}\Big|=O\left(\frac{N^{-\alpha}}{\log^{\delta-1}N}\right)=O\left(\frac{h^\alpha}{l_1(h)^{\delta-1}}\right)=O\left(\frac{h^\alpha}{l_1(h)^{\gamma(1+\varepsilon)}}\right)\leqq\frac{C_1\,h^\alpha}{[l_1(h)^{1+\varepsilon}]^\gamma}.$$

Since $s'_\nu(x)=O(\nu^{3/2})$, we find by the mean-value theorem that $s_\nu(x+h)-s_\nu(x)=O(h\nu^{3/2})$; the terms in P are therefore

$$O(h\,\nu^{3/2})\,\Delta\,\nu^{-\gamma}\log^{-\delta}\nu=O(h\,\nu^{3/2-\gamma-1}\log^{-\delta}\nu)=O(h\,\nu^{-\alpha}\log^{-\delta}\nu);$$

hence, on account of (4),

$$|P|=\Big|\sum_{1}^{N}\Big|=O\left(h\,\frac{N^{1-\alpha}}{\log^{\delta-1}N}\right)=O\left(\frac{h^\alpha}{l_1(h)^{\delta-1}}\right)\leqq\frac{C_2\,h^\alpha}{[l_1(h)^{1+\varepsilon}]^\gamma}.$$

Finally

$$|f(x+h)-f(x)|\leqq|P|+|Q|\leqq\frac{C\,h^\alpha}{[l_1(h)^{1+\varepsilon}]^\gamma}.$$

If, for any $\alpha'>\alpha$, $f(x)$ would belong to the Lipschitz-class of order α', we should have, by SZÁSZ's Theorem for the real component of $f(x)$, $\Sigma\,\varrho_n^p<\infty$ for every p satisfying

$$2/(2\alpha'+1)<p<2/(2\alpha+1)=\gamma^{-1}.$$

However, if $p<\gamma^{-1}$, the series $\Sigma\,\varrho_n^p$ diverges, since

$$\Sigma\left(\frac{1}{n^\gamma(\log n)^\delta}\right)^p$$

diverges.

G. H. HARDY [5]) has shown that if $f(x)$ satisfies a Lipschitz-condition of order $\alpha\,(0<\alpha\leqq 1)$, then

$$\sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}}\varrho_n$$

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5) G. H. HARDY, WEIERSTRASS's non-differentiable function, Tr. of the Am. Math. Soc. 17 (1916), 301—325.

converges for $\beta < \alpha$. We shall prove the following addition:

Theorem 3. *If* $0 < \alpha \leq 1$, $\varepsilon > 0$, $h > 0$, *and*

$$|f(x+h)-f(x)| \leqslant \frac{C\,h^\alpha}{l_1(h)\,l_2(h)\ldots l_k(h)^{1+\varepsilon}}, \quad . \quad . \quad . \quad . \quad (5)$$

then the series $\sum_{n=1}^{\infty} n^{\alpha-\frac{1}{2}}\,\varrho_n$ *converges.*

Proof. We shall again give the proof for $k=2$. In the same way as in Theorem 1 we find

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n^2 \leqslant \frac{C_3\,2^{-2\nu\alpha}}{[(\nu-2)\log^{1+\varepsilon}(\nu-2)]^2},$$

and from this, by SCHWARZ's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \varrho_n \leqslant \left(\Sigma\varrho_n^2\right)^{1/2}.\,2^{\nu/2} \leqslant \frac{C_4\,2^{\nu(\frac{1}{2}-\alpha)}}{(\nu-2)\log^{1+\varepsilon}(\nu-2)},$$

so that

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} n^{\alpha-\frac{1}{2}}\,\varrho_n \leqslant \frac{C_5}{(\nu-2)\log^{1+\varepsilon}(\nu-2)},$$

and finally

$$\sum_{n=2^{\nu_0-1}+1}^{\infty} n^{\alpha-\frac{1}{2}}\,\varrho_n \leqslant \sum_{\nu=\nu_0}^{\infty} \frac{C_5}{(\nu-2)\log^{1+\varepsilon}(\nu-2)} < \infty.$$

Remark. It is not difficult to prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{i\,n\log n}}{n^\gamma(\log n)^{2+\varepsilon}}\,e^{i\,n\,x}$$

where $\gamma = \dfrac{2\alpha+1}{2}$, $\varepsilon > 0$, satisfies a condition of the form (5), without belonging to a Lipschitz-class of order $\alpha' > \alpha$.

Correction, added after reading the proof-sheets. The results of (3) and (4) can be improved, since in both the right sides $\delta-1$ may be replaced by δ. For (3) this is trivial, and for (4) it follows from the splitting up of the integral into two parts, one from 2 to $N^{1/2}$ and the other from $N^{1/2}$ to N. The first integral is then

$$O(N^{\frac{1}{2}(1-\alpha)}) = O(N^{1-\alpha}\log^{-\delta}N)$$

and the second does not exceed

$$\log^{-\delta}N^{1/2}\,O(N^{1-\alpha}) = O(N^{1-\alpha}\log^{-\delta}N).$$

These results enable us to replace, in the announcement of Theorem 2, $\delta = 1+\gamma(1+\varepsilon)$ by $\delta = \gamma(1+\varepsilon)$. For the same reason we may replace, in the last remark, $2+\varepsilon$ by $1+\varepsilon$.

Mathematics. — *Sur quelques propriétés extrémales du domaine de* KOEBE. By H. BOLDER. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of October 27, 1945.)

§ 1. *Notations, définitions et conventions.*

Indiquons par

Ω	: un domaine ouvert simple et simplement connexe dans le plan d'une variable complexe, contenant dans son intérieur le point 0, et non le point ∞;
Σ	: la frontière de Ω;
$G(p, q)$	: la valeur en q de la fonction de GREEN de Ω avec pôle en p;
$\left[\dfrac{\partial G(p, z)}{\partial n}\right]_q$	: la dérivée normale en q, vers l'intérieur, donc $\geqslant 0$;
$g(p)$	: la valeur définie par le développement $G(p, p+h) = \log \lvert h \rvert^{-1} + g(p) + O(h)$, h petit;
domaine K	: domaine de KOEBE, c'est à dire un Ω, dont Σ se compose de tous les points $re^{i\theta}$, où θ est constant, et $r \geqslant d > 0$, d constant;
Ω^*, Σ^*, etc.	: des entités analogues aux précédentes, dont nous convenons, qu'elles seront réservées aux domaines K dans le plan de w;
α et β	: les deux parties de l'axe de symmétrie de l'intérieur d'un domaine K, séparées par zéro. α sera la partie entre zéro et Σ^*, β la partie infinie au côté opposé. Zéro lui-même n'appartiendra à aucune des deux classes α et β;
$\sigma(p, G)$	: l'ensemble de points z de Ω pour lesquels $G(p, z) = G$, G constant > 0. C'est donc une courbe de niveau de la fonction de GREEN;
$\varrho(p, q)$	: l'exemplaire des trajectoires orthogonales des $\sigma(p, G)$, G variable, qui passe par q.

§ 2. *Introduction.*

Parmi la classe des Ω ayants la même valeur de $g(0)$ le domaine K se distingue par quelques propriétés extrémales.

I: *la plus petite distance de* 0 *à* Σ *est minimale.* C'est donc une estimation se rapportant aux frontières des Ω. Les autres théorèmes se rapporteront aux positions et densités des courbes de niveau de leurs fonctions de GREEN.

Soient:

Ω^* un domaine K dans le plan de w, a un point de son α, et b un point de son β, telle, que

$$G^*(0, a) = G^*(0, b) = G > 0;$$

Ω un domaine dans le plan de z, pour lequel

$g(0) = g^*(0)$, et t un point de Ω pour lequel

$$G(0, t) = G^*(0, a) = G^*(0, b) = G.$$

Alors nous avons les inégalités:

$$\textbf{II:}\quad |a| \leqslant |t|$$

$$\textbf{III:}\quad |b| \left[\frac{\partial G^*(0, w)}{\partial n}\right]_b \leqslant |t| \left[\frac{\partial G(0, z)}{\partial n}\right]_t$$

$$\textbf{IV:}\quad |b| \geqslant |t|$$

$$\textbf{V:}\quad \left[\frac{\partial G^*(0, w)}{\partial n}\right]_b \leqslant \left[\frac{\partial G(0, z)}{\partial n}\right]_t$$

$$\textbf{VI:}\quad |a| \left[\frac{\partial G^*(0, w)}{\partial n}\right]_a \geqslant |t| \left[\frac{\partial G(0, z)}{\partial n}\right]_t$$

$$\textbf{VII:}\quad \left[\frac{\partial G^*(0, w)}{\partial n}\right]_a \geqslant \left[\frac{\partial G(0, z)}{\partial n}\right]_t$$

C'est par ces formules que s'expriment en termes de la fonction de GREEN des inégalités classiques sur les fonctions de la forme

$$f(z) = z + a_2 z^2 + \dots,$$

holomorphes et univalentes (simples) pour $|z| < 1$.

Dans un article précédent [1]), j'ai montré, à propos d'une suggestion de M. A. F. MONNA [2]), que le théorème de KOEBE (I) se démontre par un balayage, tout comme le théorème de CARLEMAN—MILLOUX, dans la solution élégante de M. M. BRELOT [3]).

Dans l'article présent les formules II—VII seront développées de I, suivant une méthode connue, p. e. chez M. R. NEVANLINNA [4]). Les formules II—VII correspondent aux formules 28—32 du passage cité.

§ 3. *Lemmes.*

Lemme A. *Soit un Ω_I représenté conformément sur un Ω_{II} de manière que les zéros se correspondent. Si l'on réduit Ω_I et Ω_{II} à des parties Ω'_I et Ω'_{II}, qui se correspondent dans la représentation, et qui contiennent les zéros, leurs $g(0)$ décroîtront d'une même quantité.*

Démonstration. Si Ω_I correspond à Ω_{II}, Ω'_I à Ω'_{II}, et z_I à z_{II} nous avons:

$$G_I(0, z_I) = G_{II}(0, z_{II}) \text{ et } G'_I(0, z_I) = G'_{II}(0, z_{II}),$$

donc

$$G_I(0, z_I) - G'_I(0, z_I) = G_{II}(0, z_{II}) - G'_{II}(0, z_{II}),$$

ou

$$[\log |z_I|^{-1} + g_I(0) + O(z_I)] - [\log |z_{II}|^{-1} + g'_I(0) + O(z_I)] =$$
$$= [\log |z_{II}|^{-1} + g_{II}(0) + O(z_{II})] - [\log |z_{II}|^{-1} + g'_{II}(0) + O(z_{II})].$$

En faisant tendre z_I, et donc z_{II} vers 0 nous obtenons:

$$g_I(0) - g'_I(0) = g_{II}(0) - g'_{II}(0).$$

Nous en utiliserons le cas particulier:

Si l'on retranche d'un Ω la partie d'une $\varrho(0, q)$ entre Σ et un certain niveau G de la fonction $G(0, w)$, son $g(0)$ décroîtra d'une quantité, qui ne dépend que de G.

Lemme B. *Soient p_I et q_I deux points intérieurs de Ω_I, et p_{II} et q_{II} deux points intérieurs de Ω_{II}, et soit $G_I(p_I, q_I) = G_{II}(p_{II}, q_{II})$, les expressions*

$$g(q) + \log \left(\frac{\partial G(p, w)}{\partial n}\right)_q$$

seront égales pour I *et* II.

Démonstration. Choisissons, pour I et II, un point r sur $\varrho(p, q)$ entre p et q, dans le voisinage de q, et de manière que

$$G_I(p_I, r_I) = G_{II}(p_{II}, r_{II}).$$

Nous pouvons représenter Ω_I sur Ω_{II} de manière que p_I corresponde à p_{II}, et q_I à q_{II}. Alors r_I correspondra à r_{II}.

Nous avons donc

$$G_I(q_I, r_I) = G_{II}(q_{II}, r_{II}).$$

Désignant $|q_I - r_I|$ par h_I, et $|q_{II} - r_{II}|$ par h_{II}, h_I et h_{II} petits, nous avons, pour I et II

$$[G(p, r) - G(p, q)] : h = \left(\frac{\partial G(p, z)}{\partial n}\right)_q + \varepsilon,$$

ou

$$\log [G(p, r) - G(p, q)] - \log h = \log \left(\frac{\partial G(p, z)}{\partial n}\right)_q + \varepsilon',$$

et

$$G(q, r) = \log h^{-1} + g(q) + \varepsilon''.$$

En additionnant nous éliminons h, et obtenons

$$\log [G(p, r) - G(p, q)] + G(q, r) = \log \left(\frac{\partial G(p, z)}{\partial n}\right)_q + g(q) + \varepsilon' + \varepsilon''.$$

Ces expressions sont égales pour I et II.

La considération, que les ε sont arbitraires termine la démonstration.

Remarque. Considérant un cas particulier, p.e. pour Ω le cercle $|z| < 1$, on obtient facilement la forme explicite

$$g(q) + \log\left(\frac{\partial G(p, z)}{\partial n}\right)_q = \log\left(e^{G(p,q)} - e^{-G(p,q)}\right).$$

Avant d'être en état de déduire les théorèmes II—VII, il nous faudra disposer d'une forme plus générale de I.

Par le balayage [1]) il parut sous la forme:

Parmi les Ω, pour lesquels les plus petites distances de 0 à Σ sont égales, le domaine K a la plus grande valeur de $g(0)$.

Songeant, que par une homotéthie avec centre 0 et dilatation p la valeur de $g(0)$ croisse de $\log p$, nous pouvons formuler I comme suit:

Parmi l'ensemble des Ω les domaines K ont la plus petite valeur de $\log d - g(0)$, où d désigne la plus petite distance de 0 à Σ.

§ 4. *Démonstration des théorèmes.*

II. Ajoutons dans le plan de w à Σ^* d'un Ω^* la partie de α entre a et Σ^*, et dans le plan de z à Σ d'un Ω la partie de $\varrho(0, t)$ entre t et Σ.

Les domaines restants s'appellerons $\Omega^{*\prime}$ et Ω'.

$\Omega^{*\prime}$ étant aussi un domaine K, nous savons de I

$$\log|a| - g^{*\prime}(0) \leqslant \log|t| - g'(0),$$

et du lemme A

$$g^*(0) - g^{*\prime}(0) = g(0) - g'(0).$$

En soustrayant nous obtenons

$$\log|a| - g^*(0) \leqslant \log|t| - g(0),$$

et en vertu de la supposition $g^*(0) = g(0)$ nous en tirons

$$|a| \leqslant |t|.$$

III. Dans Ω^* nous prenons pour centre b, et dans Ω pour centre t. Alors nous avons

$$G^*(b, 0) = G(t, 0),$$

et II se présente sous la forme

$$\log|b| - g^*(b) \leqslant \log|t| - g(t).$$

Nous savons de lemme B

$$g^*(b) + \log\left(\frac{\partial G^*(0, w)}{\partial n}\right)_b = g(t) + \log\left(\frac{\partial G(0, z)}{\partial n}\right)_t.$$

L'addition de ces relations termine la démonstration.

IV. Soit w un point de β entre 0 et b de Ω^*, et z dans Ω le point correspondant sur $\varrho(0, t)$, donc

$$G^*(0, w) = G(0, z).$$

Indiquons par $l(z)$ la longueur d'arc de la partie de $\varrho(0, t)$ de 0 à z, et posons $s(w) = |w|$.

En employant $l(z) \geqq |z|$ nous savons de III

$$s(w)\frac{\partial G^*(0, w)}{\partial n} \leqq l(z)\frac{\partial G(0, z)}{\partial n}.$$

Songeant, que dans Ω^* et Ω nous avons

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial s}, \quad \text{et} \quad \frac{\partial}{\partial n} = \frac{\partial}{\partial l},$$

nous pouvons écrire cette relation dans la forme

$$s(w)\frac{\partial G^*(0, w)}{\partial s} \leqq l(z)\frac{\partial G(0, z)}{\partial l}.$$

Considérons $s(w)$ et $l(z)$ sur β et $\varrho(0, t)$ comme fonctions de la variable indépendante $G \doteq G^*(0, w) = G(0, z)$.

Cela conduit à la relation

$$\frac{s}{\frac{ds}{dG}} \leqq \frac{l}{\frac{dl}{dG}}, \quad \text{ou} \quad \frac{\frac{ds}{dG}}{s} \geqq \frac{\frac{dl}{dG}}{l}.$$

Intégrons de G_0 à G, G_0 étant une valeur grande:

$$\log s - \log s_0 \geqq \log l - \log l_0.$$

En soustrayant de cette relation

$$\log s_0^{-1} + g^*(0) + \varepsilon^* = \log l_0^{-1} + g(0) + \varepsilon,$$

ce que nous obtenons en développant

$$G^*(0, w) = G(0, z),$$

nous éliminons s_0 et l_0, donc

$$\log s - g^*(0) - \varepsilon^* \geqq \log l - g(0) - \varepsilon.$$

Les ε^* et ε sont arbitraires, donc

$$\log s - g^*(0) \geqq \log l - g(0).$$

Songeant que $s(w) = |w|$, et $l(z) \geqq |z|$, nous avons

$$\log|w| - g^*(0) \geqq \log|z| - g(0),$$

et pour $G = G^*(0, b) = G(0, t)$ l'inégalité demandée.

V. C'est une conséquence immédiate de III et IV.

VI. Dans Ω^* nous prenons pour centre a, et dans Ω pour centre t. Alors nous avons $G^*(a, 0) = G(t, 0)$, et IV se présente sous la forme

$$\log|a| - g^*(a) \geqslant \log|t| - g(t).$$

Nous savons de lemme B

$$g^*(a) + \log\left(\frac{\partial G^*(0, w)}{\partial n}\right)_a = g(t) + \log\left(\frac{\partial G(0, z)}{\partial n}\right)_t.$$

L'addition de ces relations termine la démonstration.

VII. C'est une conséquence immédiate de II et VI.

§ 5. *Pour terminer nous remarquons* :

Nous savons que dans I c'est exclusivement le domaine K pour lequel la borne est atteinte. Alors nous pouvons facilement controler, que l'égalité dans aucune des formules II—VII ne se présente, que si Ω soit aussi un domaine K.

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Mathematics. — *Over de bepaaldheid der oplossingen van* $\triangle^k u = 0$. By H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 27, 1945.)

In Band II, 1, 1 der Encyclopädie der Mathematischen Wissenschaften deelt SOMMERFELD zonder bewijs en zonder litteratuurverwijzing de volgende stelling mee.

„De functie u is in het gebied binnen een gesloten kromme ondubbelzinnig bepaald, als zij in ieder punt van dat gebied voldoet aan

$$\triangle^k u = 0 \qquad (1)$$

en als aan den rand van het gebied

$$u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2} \ldots \frac{\partial^{k-1} u}{\partial n^{k-1}}$$

voorgeschreven waarden aannemen."

§ 1. Wij stellen ons voor, in het volgende een bewijs van die stelling te geven. Het is duidelijk, dat het daartoe voldoende zal zijn, te bewijzen, dat $u = 0$ de eenige oplossing is, die in het geheele gebied binnen de beschouwde kromme aan (1) voldoet en waarbij op die kromme

$$u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = \ldots \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0.$$

Wij voeren de voor de hand liggende beperking in tot functies, die in het beschouwde gebied met inbegrip van den rand doorloopende partieele afgeleiden tot die van de $2k^{\text{de}}$ orde hebben. Ook zullen wij er niet naar streven, de voorwaarden, die wij aan de gegeven kromme opleggen, zoo ruim mogelijk te houden. Wij denken ons die kromme van dien aard, dat ze kan worden voorgesteld door $x = \varphi_1(t)$, $y = \varphi_2(t)$, $a \leqq t \leqq b$, waarbij de functies $\varphi_1(t)$ en $\varphi_2(t)$ in het geheele beschouwde interval doorloopende afgeleiden hebben, waarbij in geen punt tegelijk $\varphi'_1(t) = 0$ en $\varphi'_2(t) = 0$, waarbij verder $\varphi_1(a) = \varphi_1(b)$, $\varphi_2(a) = \varphi_2(b)$, terwijl er geen paar waarden α en β te vinden is, voldoende aan $a \leqq \alpha < \beta \leqq b$, zoodanig, dat tegelijk $\varphi_1(\alpha) = \varphi_1(\beta)$ en $\varphi_2(\alpha) = \varphi_2(\beta)$. (Uit deze onderstellingen volgt, dat de kromme gesloten is, geen dubbelpunt heeft en in ieder punt een ondubbelzinnig bepaalde raaklijn heeft.) Wij kunnen dan als parameter ook de booglengte s kiezen en stellen

$$x = \psi_1(s), \quad y = \psi_2(s), \quad 0 \leqq s \leqq L.$$

Voor de functies ψ'_1 en ψ'_2 gelden dan de zelfde eigenschappen als voor φ_1 en φ_2 (met vervanging van a en b door 0 en L).

Wij hebben dan

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\psi'_1+\frac{\partial u}{\partial y}\psi'_2 \text{ en } \frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}\psi'_2-\frac{\partial u}{\partial y}\psi'_1,\ (\psi'^2_1+\psi'^2_2=1).$$

Is nu overal aan de kromme $u=0$, dus ook $\frac{\partial u}{\partial s}=0$, en $\frac{\partial u}{\partial n}=0$, dan leiden we daaruit af, dat ook $\frac{\partial u}{\partial x}=0$ en $\frac{\partial u}{\partial y}=0$.

Daaruit volgt dan weer in elk punt der kromme

$$\frac{\partial^2 u}{\partial x\,\partial s}=\frac{\partial^2 u}{\partial x^2}\psi'_1+\frac{\partial^2 u}{\partial x\partial y}\psi'_2=0,$$

en

$$\frac{\partial^2 u}{\partial y\,\partial s}=\frac{\partial^2 u}{\partial x\,\partial y}\psi'_1+\frac{\partial^2 u}{\partial y^2}\psi'_2=0.$$

Voegen we hierbij

$$\frac{\partial^2 u}{\partial n^2}=\frac{\partial^2 u}{\partial x^2}\psi'^2_2-2\frac{\partial^2 u}{\partial x\,\partial y}\psi'_1\psi'_2+\frac{\partial^2 u}{\partial y^2}\psi'^2_1=0,$$

dan hebben we drie homogene vergelijkingen, waarbij de determinant van het stelsel de waarde 1 heeft, zoodat wij kunnen besluiten, dat in ieder punt der kromme

$$\frac{\partial^2 u}{\partial x^2}=0,\ \frac{\partial^2 u}{\partial x\,\partial y}=0,\ \frac{\partial^2 u}{\partial y^2}=0.$$

Door de afgeleiden van deze functies naar s in afgeleiden van u naar x en y uit te drukken en aan de vergelijkingen, die ontstaan door deze gelijk nul te stellen, toe te voegen de vergelijking, die men verkrijgt door $\frac{\partial^3 u}{\partial n^3}=0$ in de afgeleiden $\frac{\partial^3 u}{\partial x^3}$ enz. uit te drukken, vindt men vier homogene vergelijkingen met determinant —1, waaruit men kan besluiten, dat de vier partieele afgeleiden van de derde orde in elk punt der kromme nul zijn en zoo vervolgens tot die van de $(k-1)^{\text{de}}$ orde (inclusief). Hieruit volgt natuurlijk, dat aan den rand ook $\triangle u, \triangle^2 u, \ldots \triangle^{\left[\frac{k-1}{2}\right]} u$ nul zijn.

Wij stellen nu

$$\left.\begin{array}{r}\triangle u=u_1,\ \triangle^2 u=\triangle u_1=u_2\ldots.\ \triangle^h u=\triangle^{h-1}u_1=\triangle^{h-2}u_2=\ldots u_h,\\ (h=1,2,\ldots.k-1)\end{array}\right\}\quad (2)$$

Dan is

$$\triangle u_{k-1}=0 \quad \ldots\ldots\ldots\ldots \quad (3)$$

en aan den rand,

$$\left.\begin{array}{ll}\text{voor } k \text{ even, } k=2m, & u=u_1=u_2=\ldots.u_m=0,\\ & \dfrac{\partial u}{\partial n}=\dfrac{\partial u_1}{\partial n}=\ldots.\dfrac{\partial u_{m-1}}{\partial n}=0,\\ \text{voor } k \text{ oneven, } k=2m+1, & u=u_1=u_2=\ldots.u_m=0,\\ & \dfrac{\partial u}{\partial n}=\dfrac{\partial u_1}{\partial n}=\ldots.\dfrac{\partial u_m}{\partial n}=0\end{array}\right\}. \qquad (4)$$

Wij passen nu het theorema van GREEN

$$\int (U \triangle V - V \triangle U)\, d\sigma = \int \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) ds$$

toe voor de gegeven kromme en voor $U = u$, $V = u_{k-1}$. Daarbij veronderstellen we, dat de afgeleiden der gevraagde functie u tot die van de $2k^{\text{de}}$ orde (dat zijn die welke in de vergelijking (1) voorkomen), in het geheele gebied binnen de kromme doorloopend zijn. Wij vinden

$$\int u_{k-1} \triangle u \, d\sigma = 0.$$

Dus, volgens (2)

$$\int \triangle u_{k-2}\, u_1\, d\sigma = 0. \qquad \ldots\ldots\ldots\ldots (6)$$

Nu is

$$\int (u_1 \triangle u_{k-2} - u_{k-2} \triangle u_1)\, d\sigma = \int \left(u_1 \frac{\partial u_{k-2}}{\partial n} - u_{k-2} \frac{\partial u_1}{\partial n} \right) ds = 0$$

volgens (4).
Uit (6) volgt dus

$$\int u_{k-2} \triangle u_1 \, d\sigma = 0,$$

wat wij weer met behulp van (2) herleiden. Wij vinden dan op de zelfde manier achtereenvolgens

$$\int u_{k-3} \triangle u_2 \, d\sigma = \int u_{k-4} \triangle u_3 \, d\sigma = \ldots. = 0.$$

Voor $k = 2m$, komen we zoo tot

$$\int u_m \triangle u_{m-1} \, d\sigma = 0,$$

dus

$$\int u_m^2 \, d\sigma = 0,$$

waaruit we besluiten, dat overal binnen de kromme $u_m = 0$, dus $\triangle u_{m-1} = 0$,

en daar aan den rand $u_{m-1} = 0$, ook overal binnen de kromme $u_{m-1} = 0$, dus $\triangle u_{m-2} = 0$, zoo doorgaande komen we tot $u = 0$ overal binnen de kromme.

Voor $k = 2m + 1$ voeren de voorgaande herleidingen tot

$$\int u_m \triangle u_m \, d\sigma = 0,$$

waaruit in verband met (4)

$$\int \left\{ \left(\frac{\partial u_m}{\partial x} \right)^2 + \left(\frac{\partial u_m}{\partial y} \right)^2 \right\} d\sigma = 0.$$

Hieruit besluiten we, dat overal binnen de kromme $\dfrac{\partial u_m}{\partial x} = 0$ en $\dfrac{\partial u_m}{\partial y} = 0$, dus u_m in het geheele gebied constant en daar het aan den rand nul is, is het overal nul. De rest van het bewijs gaat weer evenals in het vorige geval.

§ 2. Wij zullen nu de analoge stelling bewijzen voor het geval, dat $\triangle$ wordt opgevat als een symbool in drie onafhankelijk veranderlijken. Daarbij beschouwen we een oppervlak, dat kan worden voorgesteld door

$x = \varphi_1 (s_1 \, t)$, $y = \varphi_2 (s, t)$, $x = \varphi_3 (s, t)$, $a_1 \leqq s \leqq b_1$, $a_2 \leqq t \leqq b_2$,

waarbij de functies $\varphi_1, \varphi_2, \varphi_3$ in het geheele beschouwde gebied doorloopende partieele afgeleiden hebben, waarbij in geen punt tegelijk

$$\begin{pmatrix} y\,z \\ s\,t \end{pmatrix} = 0, \begin{pmatrix} z\,x \\ s\,t \end{pmatrix} = 0 \text{ en } \begin{pmatrix} x\,y \\ s\,t \end{pmatrix} = 0,$$

waarbij verder voor iedere t uit het beschouwde interval

$$\varphi_1 (a_1, t) = \varphi_1 (b_1, t), \varphi_2 (a_1, t) = \varphi_2 (b_1, t) \text{ en } \varphi_3 (a_1, t) = \varphi_3 (b_1, t)$$

en voor iedere s uit het beschouwde interval

$$\varphi_1 (s, a_2) = \varphi_1 (s, b_2), \varphi_2 (s, a_2) = \varphi_2 (s, b_2) \text{ en } \varphi_3 (s, a_3) = \varphi_3 (s, b_3),$$

terwijl er geen waardenparen α_1, β_1 en α_2, β_2, voldoende aan

$$a_1 \leqq \alpha_1 < \beta_1 \leqq b_1 \text{ en } a_2 \leqq \alpha_2 < \beta_2 \leqq b_2$$

te vinden zijn zoodanig, dat te gelijk

$$\varphi_1 (\alpha_1, \alpha_2) = \varphi_1 (\beta_1, \beta_2), \varphi_2 (\alpha_1, \alpha_2) = \varphi_2 (\beta_1, \beta_2) \text{ en } \varphi_3 (\alpha_1, \alpha_2) = \varphi_3 (\beta_1, \beta_2).$$

Het oppervlak is dus gesloten, zonder dubbelpunt en heeft in ieder punt een ondubbelzinnig bepaald raakvlak.

Wij willen nu vooreerst bewijzen, dat als een functie u overal op het oppervlak nul is en bovendien overal op het oppervlak $\dfrac{\partial u}{\partial n} = 0$, dan zijn

van die functie in ieder punt van het oppervlak alle eerste afgeleiden gelijk aan nul; is bovendien in ieder punt $\frac{\partial^2 u}{\partial n^2} = 0$, dan zijn ook alle tweede afgeleiden gelijk nul enz.

Daar overal aan het oppervlak $u = 0$, is ook $\frac{\partial u}{\partial s} = 0$ en $\frac{\partial u}{\partial t} = 0$. Daaruit volgt:

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} = 0,$$

en

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = 0,$$

dus

$$\frac{\frac{\partial u}{\partial x}}{\begin{pmatrix} y\,z \\ s\,t \end{pmatrix}} = \frac{\frac{\partial u}{\partial y}}{\begin{pmatrix} z\,x \\ s\,t \end{pmatrix}} = \frac{\frac{\partial u}{\partial z}}{\begin{pmatrix} x\,y \\ s\,t \end{pmatrix}} = \lambda_1 .$$

De voorwaarde $\frac{\partial u}{\partial n} = 0$ geeft

$$\frac{\partial u}{\partial x}\begin{pmatrix} y\,z \\ s\,t \end{pmatrix} + \frac{\partial u}{\partial y}\begin{pmatrix} z\,x \\ s\,t \end{pmatrix} + \left(\frac{\partial u}{\partial z}\right)\begin{pmatrix} x\,y \\ s\,t \end{pmatrix} = 0,$$

dus

$$\lambda_1 \left\{ \begin{pmatrix} y\,z \\ s\,t \end{pmatrix}^2 + \begin{pmatrix} z\,x \\ s\,t \end{pmatrix}^2 + \begin{pmatrix} x\,y \\ s\,t \end{pmatrix}^2 \right\} = 0,$$

waaruit $\lambda_1 = 0$ en dus alle eerste partieele afgeleiden van u gelijk nul. Daar $\frac{\partial u}{\partial x}$ overal op het oppervlak de waarde nul heeft is ook

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) = 0 \text{ en } \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = 0,$$

hetgeen geeft

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial s} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial z}{\partial s} = 0,$$

en

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial y}{\partial t} + \frac{\partial^2 u}{\partial x \partial z}\frac{\partial z}{\partial t} = 0,$$

dus

$$\frac{\dfrac{\partial^2 u}{\partial x^2}}{\begin{pmatrix} y & z \\ s & t \end{pmatrix}} = \frac{\dfrac{\partial^2 u}{\partial x \partial y}}{\begin{pmatrix} z & x \\ s & t \end{pmatrix}} = \frac{\dfrac{\partial^2 u}{\partial x \partial z}}{\begin{pmatrix} x & y \\ s & t \end{pmatrix}} = \mu_1,$$

evenzoo

$$\frac{\dfrac{\partial^2 u}{\partial x \partial y}}{\begin{pmatrix} y & z \\ s & t \end{pmatrix}} = \frac{\dfrac{\partial^2 u}{\partial y^2}}{\begin{pmatrix} z & x \\ s & t \end{pmatrix}} = \frac{\dfrac{\partial^2 u}{\partial y \partial z}}{\begin{pmatrix} x & y \\ s & t \end{pmatrix}} = \mu_2,$$

en

$$\frac{\dfrac{\partial^2 u}{\partial x \partial z}}{\begin{pmatrix} y & z \\ s & t \end{pmatrix}} = \frac{\dfrac{\partial^2 u}{\partial y \partial z}}{\begin{pmatrix} z & x \\ s & t \end{pmatrix}} = \frac{\dfrac{\partial^2 u}{\partial z^2}}{\begin{pmatrix} x & z \\ s & t \end{pmatrix}} = \mu_3,$$

waaruit

$$\frac{\mu_1}{\begin{pmatrix} y & z \\ s & t \end{pmatrix}} = \frac{\mu_2}{\begin{pmatrix} z & x \\ s & t \end{pmatrix}} = \frac{\mu_3}{\begin{pmatrix} x & y \\ s & t \end{pmatrix}} = \lambda_2$$

dus

$$\frac{\partial^2 u}{\partial x^2} = \lambda_2 \begin{pmatrix} y & z \\ s & t \end{pmatrix}^2, \quad \frac{\partial^2 u}{\partial x \partial y} = \lambda_2 \begin{pmatrix} y & z \\ s & t \end{pmatrix} \begin{pmatrix} z & x \\ s & t \end{pmatrix} \text{ enz.}$$

De voorwaarde $\dfrac{\partial^2 u}{\partial n^2} = 0$ geeft

$$\frac{\partial^2 u}{\partial x^2} \begin{pmatrix} y & z \\ s & t \end{pmatrix}^2 + 2 \frac{\partial^2 u}{\partial x \partial y} \begin{pmatrix} y & z \\ s & t \end{pmatrix} \begin{pmatrix} z & x \\ s & t \end{pmatrix} + 2 \frac{\partial^2 u}{\partial x \partial z} \begin{pmatrix} y & z \\ s & t \end{pmatrix} \begin{pmatrix} x & y \\ s & t \end{pmatrix} +$$

$$+ \frac{\partial^2 u}{\partial y^2} \begin{pmatrix} z & x \\ s & t \end{pmatrix}^2 + 2 \frac{\partial^2 u}{\partial y \partial z} \begin{pmatrix} z & x \\ s & t \end{pmatrix} \begin{pmatrix} x & y \\ s & t \end{pmatrix} + \frac{\partial^2 u}{\partial z^2} \begin{pmatrix} x & y \\ s & t \end{pmatrix}^2 = 0$$

dus

$$\lambda_2 \left\{ \begin{pmatrix} y & z \\ s & t \end{pmatrix}^2 + \begin{pmatrix} z & x \\ s & t \end{pmatrix}^2 + \begin{pmatrix} x & y \\ s & t \end{pmatrix}^2 \right\}^2 = 0,$$

waaruit $\lambda_2 = 0$ en dus alle tweede afgeleiden van u aan het oppervlak gelijk aan nul.

Op de zelfde manier geeft de voorwaarde $\dfrac{\partial^3 u}{\partial n^3} = 0$ aan het oppervlak

$$\lambda_3 \left\{ \begin{pmatrix} y & z \\ s & t \end{pmatrix}^2 + \begin{pmatrix} z & x \\ s & t \end{pmatrix}^2 + \begin{pmatrix} x & y \\ s & t \end{pmatrix}^2 \right\}^3 = 0, \text{ als } \frac{\partial^3 u}{\partial x^3} = \lambda_3 \begin{pmatrix} y & z \\ s & t \end{pmatrix}^3 \text{ enz.,}$$

dus $\lambda_3 = 0$, waaruit weer volgt, dat aan het oppervlak alle partieele afgeleiden van de derde orde gelijk nul zijn, enz.

Het bewijs, dat $u = 0$ de eenige oplossing is van $\triangle^k u = 0$, waarbij in ieder punt van het oppervlak

$$u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = \ldots . \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0,$$

verloopt nu geheel als het vorige. Men heeft eenvoudig in de formules (5) en volgende de oppervlakte-integralen door ruimte-integralen en de integralen over de randkromme door integralen over het grensoppervlak te vervangen.

De uitbreiding tot het geval, dat $\triangle u$ een symbool voorstelt in n onafhankelijk veranderlijken geschiedt nu op de zelfde wijze.

Mathematics. — *Eigenschappen der oplossingen van* $\Delta^k = 0$. By H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 27, 1945.)

§ 1. Wij willen bewijzen, dat, als in zeker gebied $\Delta u = 0$, waarbij Δ den operator van LAPLACE in twee onafhankelijk veranderlijken voorstelt en u een functie is, waarvan de partiëele afgeleiden tot en met die van de $2\,k$de orde bestaan, dan is in hetzelfde gebied $\Delta^k r^{2k-2}\, u = 0$, waarbij r den afstand voorstelt tot een willekeurig punt, dat wij als oorsprong van coördinaten kiezen.

Wij beginnen met een paar hulpstellingen.

I. Als $\Delta u = 0$, is ook $\Delta r \dfrac{\partial u}{\partial r} = 0$.

Bewijs.

$$r\,\frac{\partial u}{\partial r} = x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y},$$

$$\Delta\, x\,\frac{\partial u}{\partial x} = x\,\frac{\partial \Delta u}{\partial x} + 2\,\frac{\partial^2 u}{\partial x^2},\ \Delta\, y\,\frac{\partial u}{\partial y} = y\,\frac{\partial \Delta u}{\partial y} + 2\,\frac{\partial^2 u}{\partial y^2},$$

dus $\Delta\, r\,\dfrac{\partial u}{\partial r} = x\,\dfrac{\partial \Delta u}{\partial x} + y\,\dfrac{\partial \Delta u}{\partial y} + 2\,\Delta\, u$, dus als $\Delta\, u = 0$, $\Delta\, r\,\dfrac{\partial u}{\partial r} = 0$.

II. $\Delta\, r^{2k}\, u = r^{2k}\,\Delta\, u + 4\,k\,r^{2k-1}\,\dfrac{\partial u}{\partial r} + 4\,k^2\,r^{2k-2}\,u.$

Bewijs. De betrekking geldt voor $k = 1$, immers

$$\frac{\partial^2}{\partial x^2}\,(x^2 u) = x^2\,\frac{\partial^2 u}{\partial x^2} + 4x\,\frac{\partial u}{\partial x} + 2u,$$

dus

$$\frac{\partial^2}{\partial x^2}\,(r^2 u) = r^2\,\frac{\partial^2 u}{\partial x^2} + 4\,x\,\frac{\partial u}{\partial x} + 2\,u.$$

Evenzoo

$$\frac{\partial^2}{\partial y^2}\,(r^2 u) = r^2\,\frac{\partial^2 u}{\partial y^2} + 4\,y\,\frac{\partial u}{\partial y} + 2\,u,$$

dus

$$\Delta\, r^2\, u = r^2\,\Delta\, u + 4\,r\,\frac{\partial u}{\partial r} + 4u. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Als ze geldt voor zeker natuurlijk getal k, dan is (volgens de laatste formule)

$$\triangle r^{2k+2} u = r^2 \triangle r^{2k} u + 4 r \frac{\partial}{\partial r}(r^{2k} u) + 4 r^{2k} u$$

$$= r^{2k+2} \triangle u + 4kr^{2k+1} \frac{\partial u}{\partial r} + 4 k^2 r^{2k} u + 4 r^{2k+1} \frac{\partial u}{\partial r} + 8 k r^{2k} u + 4 r^{2k} u$$

$$= r^{2k+2} \triangle u + 4(k+1) r^{2k+1} \frac{\partial u}{\partial r} + 4(k+1)^2 r^{2k} u.$$

De betrekking geldt dan dus ook voor $k+1$.

De hoofdstelling bewijzen we nu ook door volledige inductie. Gegeven zij $\triangle u = 0$ en wij nemen aan, dat bewezen is, dat dan $\triangle^k r^{2k-2} u = 0$, dan is, met toepassing van hulpstelling II

$$\triangle^{k+1} r^{2k} u = \triangle^k (\triangle r^{2k} u) = 4 k \triangle^k r^{2k-2} \left(r \frac{\partial u}{\partial r} \right) + 4k^2 \triangle^k r^{2k-2} u.$$

Nu is $\triangle r \dfrac{\partial u}{\partial r} = 0$ volgens hulpstelling I, dus volgens het bij de volledige inductie ondersteld $\triangle^k r^{2k-2} \left(r \dfrac{\partial u}{\partial r} \right) = 0$ en ook $\triangle^k r^{2k-2} u = 0$, dus $\triangle^{k+1} r^{2k} u = 0$, waarmee de bovengenoemde stelling bewezen is.

§ 2. Wij zullen nu verder bewijzen, dat iedere oplossing der vergelijking $\triangle^k u = 0$ kan gebracht worden in de gedaante

$$u = r^{2k-2} u_0 + r^{2k-4} u_1 + r^{2k-6} u_2 + \ldots + r^2 u_{k-2} + u_{k-1}, \quad . \quad (2)$$

waarbij $u_0, u_1, \ldots u_{k-1}$ harmonische functies zijn.

Deze stelling geldt voor $k = 1$. Om tot een bewijs door volledige inductie te geraken, is het voldoende te bewijzen, dat, als $\triangle^{k+1} u = 0$, er twee functies u_0 en v te vinden zijn zoo, dat

$$u = r^{2k} u_0 + v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

met $\triangle u_0 = 0$ en $\triangle^k v = 0$. Dan moet dus $\triangle^k r^{2k} u_0 = \triangle^k u = \alpha$, waarbij α een gegeven functie is zoodanig, dat $\triangle \alpha = 0$.

Wij bewijzen nu vooreerst: als $\triangle \varphi = 0$, geldt voor een willekeurig natuurlijk getal h:

$$\triangle^h r^{2h} \varphi = h!\, 2^{2h} \left(r \frac{\partial}{\partial r} + h \right) \left(r \frac{\partial}{\partial r} + h - 1 \right) \ldots \left(r \frac{\partial}{\partial r} + 1 \right) \varphi. \quad . \quad (4)$$

Deze betrekking geldt volgens hulpstelling II voor $h = 1$. Geldt ze voor

een natuurlijk getal h, dan hebben we ook, onder toepassing van die zelfde stelling

$$\triangle^{h+1} r^{2h+2} \varphi = \triangle^h \triangle r^{2h+2} \varphi =$$

$$= \triangle^h \left\{ r^{2h+2} \triangle \varphi + 4(h+1) r^{2h+1} \frac{\partial \varphi}{\partial r} + 4(h+1)^2 r^{2h} \varphi \right\} =$$

$$= 4(h+1) \left\{ \triangle^h r^{2h} \left(r \frac{\partial \varphi}{\partial r} \right) + (h+1) \triangle^h r^{2h} \varphi \right\}$$

en dus door toepassing der formule, die bij de volledige inductie als bewezen is aangenomen, daar volgens hulpstelling I ook $\triangle r \dfrac{\partial \varphi}{\partial r} = 0$,

$$\triangle^{h+1} r^{2h+2} \varphi = (h+1)!\, 2^{2h+2} \left\{ \left(r \frac{\partial}{\partial r} + h \right) \left(r \frac{\partial}{\partial r} + h - 1 \right) \cdots \left(r \frac{\partial}{\partial r} + 1 \right) r \frac{\partial}{\partial r} \varphi \right.$$

$$\left. + (h+1) \left(r \frac{\partial}{\partial r} + h \right) \cdots \left(r \frac{\partial}{\partial r} + 1 \right) \varphi \right\},$$

dus

$$\triangle^{h+1} r^{2h+2} \varphi = (h+1)!\, 2^{2(h+1)} \left(r \frac{\partial}{\partial r} + h + 1 \right) \left(r \frac{\partial}{\partial r} + h \right) \cdots \left(r \frac{\partial}{\partial r} + 1 \right) \varphi .$$

De betrekking geldt dan dus ook voor $h+1$.

De toepassing op ons geval geeft

$$\triangle^k r^{2k} u_0 = k!\, 2^{2k} \left(r \frac{\partial}{\partial r} + k \right) \left(r \frac{\partial}{\partial r} + k - 1 \right) \cdots \left(r \frac{\partial}{\partial r} + 1 \right) u_0 = \alpha . \quad (5)$$

Wij kunnen deze differentiaalvergelijking in den vorm brengen

$$\frac{\partial^k}{\partial r^k} (r^k u_0) = \frac{1}{k!\, 2^{2k}} \alpha . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hieraan voldoet

$$u_0 = \frac{1}{k!\, 2^{2k}\, r^k} \int_0^r d\varrho_k \int_0^{\varrho_k} d\varrho_{k-1} \int_0^{\varrho_{k-1}} \cdots \int_0^{\varrho_3} d\varrho_2 \int_0^{\varrho_2} \alpha\, d\varrho_1 .$$

Door verandering der integratievolgorde herleiden wij dat tot

$$u_0 = \frac{1}{k!\,(k-1)!\, 2^{2k}\, r^k} \int_0^r \alpha (r - \varrho_1)^{k-1}\, d\varrho_1 . \quad . \quad . \quad . \quad . \quad (7)$$

Wij willen nu bewijzen, dat voor de zoo bepaalde u_0 geldt $\triangle u_0 = 0$, verder volgt uit de manier, waarop u_0 verkregen is, dat $\triangle^k r^{2k} u_0 = \alpha$.

zoodat, als wij stellen $v = u - r^{2k} u_0$, inderdaad $\Delta^k v = 0$. Voor het bewijs, dat $\Delta u_0 = 0$ speelt een constante factor geen rol. We beschouwen daarom

$$u = \frac{1}{r^k} \int_0^r \alpha (r-\varrho)^{k-1} \, d\varrho,$$

en berekenen Δu met behulp van poolcoördinaten. Wij vinden

$$\frac{\partial u}{\partial r} = -\frac{k}{r^{k+1}} \int_0^r \alpha (r-\varrho)^{k-1} \, d\varrho + \frac{k-1}{r^k} \int_0^r \alpha (r-\varrho)^{k-2} \, d\varrho,$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{k\,(k+1)}{r^{k+2}} \int_0^r \alpha (r-\varrho)^{k-1} \, d\varrho - \frac{2\,k\,(k-1)}{r^{k+1}} \int_0^r \alpha (r-\varrho)^{k-2} \, d\varrho +$$

$$+ \frac{(k-1)(k-2)}{r^k} \int_0^r \alpha (r-\varrho)^{k-3} \, d\varrho,$$

$$\frac{\partial^2 u}{\partial \vartheta^2} = \frac{1}{r^k} \int_0^r \frac{\partial^2 \alpha}{\partial \vartheta^2} (r-\varrho)^{k-1} \, d\varrho.$$

Daar $\Delta \alpha = 0$, is $\dfrac{\partial^2 \alpha}{\partial \vartheta^2} = -\left(\varrho^2 \dfrac{\partial^2 \alpha}{\partial \varrho^2} + \varrho \dfrac{\partial \alpha}{\partial \varrho}\right) = -\varrho \dfrac{\partial}{\partial \varrho}\left(\varrho \dfrac{\partial \alpha}{\partial \varrho}\right)$,

dus

$$\frac{\partial^2 u}{\partial \vartheta^2} = -\frac{1}{r^k} \int_0^r (r-\varrho)^{k-1} \varrho \frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial \alpha}{\partial \varrho}\right) d\varrho =$$

$$\frac{1}{r^k} \int_0^r \{-(k-1)(r-\varrho)^{k-2} \varrho + (r-\varrho)^{k-1}\} \varrho \frac{\partial \alpha}{\partial \varrho} \, d\varrho =$$

$$-\frac{1}{r^k} \int_0^r (r-\varrho)^{k-2} (k\varrho^2 - r\varrho) \frac{\partial \alpha}{\partial \varrho} \, d\varrho = \frac{1}{r^k} \int_0^r \{-(k-2)(r-\varrho)^{k-3} (k\varrho^2 - r\varrho) +$$

$$+ (r-\varrho)^{k-2} (2k\varrho - r)\} \alpha \, d\varrho = \frac{1}{r^k} \int_0^r \{-k^2 \varrho^2 + (3k-1) r\varrho - r^2\} (r-\varrho)^{k-3} \alpha \, d\varrho.$$

Dus

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \vartheta^2} =$$

$$= \frac{1}{r^{2k+2}} \int_0^r (r-\varrho)^{k-3} \alpha \{k^2 (r-\varrho)^2 - (k-1)(2k-1) r (r-\varrho) + (k-1)(k-2) r^2 -$$

$$- k^2 \varrho^2 + (3k-1) r\varrho - r^2\} \, d\varrho,$$

$$\Delta u = 0.$$

Deze stellingen openen een weg voor het bewijs van het bestaan van een oplossing der vergelijking $\triangle^k u = 0$ in het gebied binnen een gesloten kromme, die met haar $k-1$ eerste normale afgeleiden aan die kromme gegeven waarden aanneemt, door deze zaak terug te brengen tot het bewijs van het bestaan van een stelsel van k functies, die in het gebied binnen die kromme harmonisch zijn, en continue particeele afgeleiden hebben tot inclusief die van de $2k$de orde, terwijl aan de kromme voldaan wordt aan k lineaire betrekkingen tusschen die functies en haar $k-1$ eerste normale afgeleiden.

§ 3. Om de analoge stellingen te bewijzen, voor het geval, dat het symbool $\triangle$ den operator van LAPLACE in drie onafhankelijk veranderlijken voorstelt, hebben we in het vorige slechts weinig te veranderen.

Hulpstelling I: Als $\triangle u = 0$, is ook $\triangle r \dfrac{\partial u}{\partial r} = 0$, blijft gelden.

Hulpstelling II luidt nu:

$$\triangle r^{2k} u = r^{2k} \triangle u + 4 k r^{2k-1} \frac{\partial u}{\partial r} + 2 k (2k+1) r^{2k-2} u. \qquad (8)$$

Om die te bewijzen, hebben we

$$\frac{\partial^2}{\partial x^2}(x^2 u) = x^2 \frac{\partial^2 u}{\partial x^2} + 4 x \frac{\partial u}{\partial x} + 2 u,$$

dus

$$\frac{\partial^2}{\partial x^2}(r^2 u) = r^2 \frac{\partial^2 u}{\partial x^2} + 4 x \frac{\partial u}{\partial x} + 2 u.$$

Evenzoo

$$\frac{\partial^2}{\partial y^2}(r^2 u) = r^2 \frac{\partial^2 u}{\partial y^2} + 4 y \frac{\partial u}{\partial y} + 2 u,$$

$$\frac{\partial^2}{\partial z^2}(r^2 u) = r^2 \frac{\partial^2 u}{\partial z^2} + 4 z \frac{\partial u}{\partial z} + 2 u,$$

dus

$$\triangle (r^2 u) = r^2 \triangle u + 4 r \frac{\partial u}{\partial r} + 6 u.$$

De betrekking (8) geldt dus voor $k = 1$. Geldt ze voor een natuurlijk getal k, dan is

$$\triangle r^{2k+2} u = \triangle r^2 (r^{2k} u) = r^{2k+2} \triangle u + 4k\, r^{2k+1} \frac{\partial u}{\partial r} + 2 k (2k+1) r^{2k} u + 4\, r^{2k+1} \frac{\partial u}{\partial r} +$$

$$+ 8 k r^{2k} u + 6 r^{2k} u = r^{2k+2} \triangle u + 4 (k+1) r^{2k+1} \frac{\partial u}{\partial r} + (2k+2)(2k+3) r^{2k} u,$$

en geldt de betrekking (8) dus ook voor $k+1$; daaruit volgt, dat ze voor ieder natuurlijk getal geldt.

Om nu te bewijzen, dat, als $\triangle u = 0$, ook $\triangle^k r^{2k-2} u = 0$, merken we op, dat als $\triangle u = 0$, ook $\triangle r \dfrac{\partial u}{\partial r} = 0$ en wij hebben, bewezen aannemende, dat $\triangle^k r^{2k-2} u = 0$,

$$\triangle^{k+1} r^{2k} u = \triangle^k \left(r^{2k} \triangle u + 4\,k\, r^{2k-1} \frac{\partial u}{\partial r} + 2\,k\,(2\,k+1)\, r^{2k-2}\, u \right) =$$

$$4\,k\, \triangle^k r^{2k-2} \left(r \frac{\partial u}{\partial r} \right) + 2\,k\,(2\,k+1)\, \triangle^k r^{2k-2} u = 0,$$

waaruit volgt, dat de betrekking voor ieder natuurlijk getal k geldt.

§ 4. Verder bewijzen we, dat ook nu, als $\triangle^{k+1} u = 0$, er twee functies u_0 en v te vinden zijn zoo, dat

$$u = r^{2k} u_0 + v, \qquad (9)$$

en dat $\triangle u_0 = 0$ en $\triangle^k v = 0$.

Dan voldoet dus u_0 aan

$$\triangle^k r^{2k} u_0 = a,$$

waarbij $a = \triangle^k u$ en, dus $\triangle a = 0$.

Nu geldt, als $\triangle \varphi = 0$, voor een natuurlijk getal h

$$\triangle^h r^{2h} \varphi = h\,!\, 2^h \left(2\,r \frac{\partial}{\partial r} + 2\,h + 1 \right) \left(2\,r \frac{\partial}{\partial r} + 2\,h - 1 \right) \ldots .$$
$$\ldots . \left(2r \frac{\partial}{\partial r} + 3 \right) \varphi. \qquad (10)$$

Immers volgens het bovenstaande geldt deze betrekking voor $h = 1$; geldt ze voor een natuurlijk getal h, dan is

$$\triangle^{h+1} r^{2h+2} \varphi = \triangle^h \triangle r^{2h+2} \varphi =$$

$$= \triangle^h \left\{ r^{2h+2} \triangle \varphi + 4\,(h+1)\, r^{2h+1} \frac{\partial \varphi}{\partial r} + (2h+2)(2h+3)\, r^{2h} \varphi \right\} =$$

$$= 4\,(h+1)\, \triangle^h r^{2h} \left(r \frac{\partial \varphi}{\partial r} \right) + (2h+2)(2h+3)\, \triangle^h r^{2h} \varphi =$$

$$h\,!\, 2^h \Big\{ 4\,(h+1) \left(2\,r \frac{\partial}{\partial r} + 2\,h + 1 \right) \left(2\,r \frac{\partial}{\partial r} + 2h - 1 \right) \ldots \left(2\,r \frac{\partial}{\partial r} + 3 \right) r \frac{\partial}{\partial r} \varphi$$

$$+ (2h+2)(2\,h+3) \left(2\,r \frac{\partial}{\partial r} + 2\,h + 1 \right) \ldots . \left(2\,r \frac{\partial}{\partial r} + 3 \right) \varphi \Big\} =$$

$$(h+1)\,!\, 2^{h+1} \left(2\,r \frac{\partial}{\partial r} + 2\,h + 3 \right) \left(2\,r \frac{\partial}{\partial r} + 2\,h + 1 \right) \ldots . \left(2\,r \frac{\partial}{\partial r} + 3 \right) \varphi$$

en (10) geldt dus ook voor $h + 1$.

Wij vinden dus

$$\Delta^k r^{2k} u_0 = k!\,2^k \left(2r\frac{\partial}{\partial r} + 2k + 1\right)\left(2r\frac{\partial}{\partial r} + 2k - 1\right)\ldots$$
$$\ldots\left(2r\frac{\partial}{\partial r} + 3\right) u_0 = \alpha. \quad (11)$$

Deze differentiaalvergelijking brengen we in den vorm

$$\frac{\partial^k}{\partial r^k}\left(r^{k+\frac{1}{2}} u_0\right) = \frac{1}{k!\,2^{2k}}\,\alpha\, r^{\frac{1}{2}}.$$

Daaraan voldoet

$$u_0 = \frac{1}{k!\,2^{2k}\, r^{k+\frac{1}{2}}} \int_0^r d\varrho_k \int_0^{\varrho_k} d\varrho_{k-1} \ldots \int_0^{\varrho_2} \alpha\, \varrho_1^{\frac{1}{2}}\, d\varrho_1 =$$
$$= \frac{1}{k!\,(k-1)!\,2^{2k}\, r^{k+\frac{1}{2}}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-1}\, d\varrho.$$

Wij hebben te bewijzen, dat ook nu voor de zoo bepaalde functie u_0 uit $\Delta\alpha = 0$ volgt $\Delta u_0 = 0$. Bij dit bewijs speelt een constante factor weer geen rol. We beschouwen dus

$$u = \frac{1}{r^{k+\frac{1}{2}}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-1}\, d\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

en berekenen Δu met behulp van bolcoördinaten (r, φ, ϑ). Wij vinden:

$$r^2 \frac{\partial u}{\partial r} = -\frac{k+{}^1/_2}{r^{k-\frac{1}{2}}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-1}\, d\varrho + \frac{k-1}{r^{k-\frac{3}{2}}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-2}\, d\varrho.$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{k^2-{}^1/_4}{r^{k+\frac{1}{2}}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-1}\, d\varrho - \frac{(k-1)(2k-1)}{r^{k-\frac{1}{2}}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-2}\, d\varrho +$$
$$+ \frac{(k-1)(k-2)}{r^{k-{}^3/_2}} \int_0^r \alpha\, \varrho^{\frac{1}{2}} (r-\varrho)^{k-3}\, d\varrho.$$

$$\frac{1}{\sin\varphi} \frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial u}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi} \frac{\partial^2 u}{\partial\vartheta^2} =$$
$$= \frac{1}{r^{k+\frac{1}{2}}} \int_0^r \varrho^{\frac{1}{2}} (r-\varrho)^{k-1} \left\{ \frac{1}{\sin\varphi} \frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial\alpha}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi} \frac{\partial^2\alpha}{\partial\vartheta^2} \right\} d\varrho.$$

Daar $\triangle \alpha = 0$, kunnen we voor de laatste integraal schrijven

$$-\frac{1}{r^{k+\frac{1}{2}}}\int_0^r \varrho^{\frac{1}{2}}(r-\varrho)^{k-1}\frac{\partial}{\partial\varrho}\left(\varrho^2\frac{\partial\alpha}{\partial\varrho}\right)d\varrho$$

en door achtereenvolgend partieel integreeren herleiden we dat tot

$$\frac{1}{r^{k+\frac{1}{2}}}\int_0^r \left\{\tfrac{1}{2}\varrho^{-1/2}(r-\varrho)^{k-1}-(k-1)\varrho^{1/2}(r-\varrho)^{k-2}\right\}\varrho^2\frac{\partial\alpha}{\partial\varrho}\,d\varrho=$$

$$-\frac{1}{r^{k+\frac{1}{2}}}\int_0^r \left\{\tfrac{3}{4}\varrho^{1/2}(r-\varrho)^{k-1}-3(k-1)\varrho^{3/2}(r-\varrho)^{k-2}+(k-1)(k-2)\varrho^{5/2}(r-\varrho)^{k-3}\right\}\alpha\,d\varrho.$$

Wij vinden dus

$$r^2\triangle u=\frac{1}{r^{k+1/2}}\int_0^r \alpha\,\varrho^{1/2}(r-\varrho)^{k-3}\left\{(k^2-\tfrac{1}{4})(r-\varrho)^2-(k-1)(2k-1)\,r\,(r-\varrho)+\right.$$
$$\left.+(k-1)(k-2)\,r^2\right\}d\varrho=$$

$$-\frac{1}{r^{k+1/2}}\int_0^r \alpha\,\varrho^{1/2}(r-\varrho)^{k-3}\left\{\tfrac{3}{4}(r-\varrho)^2-3(k-1)\varrho(r-\varrho)+(k-1)(k-2)\varrho^2\right\}d\varrho=0.$$

Door de gevonden uitdrukking voor u_0 nu in (9) in te voeren komt de willekeurige oplossing u van $\triangle^{k+1}u=0$ dus in den vorm

$$u=r^{2k}\,u_0+v,$$

waarbij $\triangle u_0=0$ en $\triangle^k v=0$. Voor v kan dan weer geschreven worden

$$v=r^{2k-2}\,u_1+v_1,$$

waarbij $\triangle u_1=0$ en $\triangle^{k-1}v_1=0$ enz.

Hiermee is bewezen, dat ook voor dit geval iedere oplossing van $\triangle^k u=0$ kan gebracht worden in den vorm

$$u=r^{2k-2}\,u_0+r^{2k-4}\,u_1+r^{2k-6}\,u_2+\ldots+r^2\,u_{k-2}+u_{k-1}.$$

De uitbreiding tot het geval, dat $\triangle$ een symbool in n onafhankelijk veranderlijken voorstelt brengt geen principieele moeilijkheden mee. Ook in deze gevallen openen de bewezen stellingen een weg voor het bewijs der overeenkomstige existentietheorema's.

Mathematics. — *Topological classification of all closed countable and continuous classification of all countable pointsets.* By J. DE GROOT. (Communicated by Prof. J. G. VAN DER CORPUT).

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Dedicated to the late Prof. Dr. G. SCHAAKE.

I.

1. 1. The problem of determining the existence or not-existence of a homoeomorphism between two spaces (pointsets) is often called the chief problem of topology. In other words one has to enumerate, if possible, all classes of not-homoeomorphic sets. In this generality the problem is by no means solved. Only for special families of sets [1]) a solution is known. We are giving some instances. The closed surfaces (closed 2-dimensional manifolds) have been completely classified topologically (compare for instance SEIFERT–THRELFALL, Lehrbuch der Topologie, Chpt. VI), for the closed 3-dimensional manifolds however the problem is as yet unsolved. For the perfect (i.e. compact, dense in themselves) 0-dimensianal sets the problem has been solved, for every two sets of that kind are homoeomorphic as both are homoeomorphic with the discontinuum D of CANTOR (Theorem of BROUWER); for the countable sets the problem of homoeomorphism has not been solved, and so of course not for the 0-dimensional sets in general. Only a theorem of SIERPIŃSKI is known, saying that every two countable sets, dense in themselves, are homoeomorphic.

We shall first solve the problem of homoeomorphism for compact countable sets (1.2.—1.6.). After that a generalisation will prove possible for the more general class of microcompact countable sets and therefore especially for the closed countable sets.

1. 2. We consider an arbitrary compact countable set A as a subset of an n-dimensional Euclidian space R_n (or as a subset of the Hilbert space R_ω). Moreover it is possible to consider A as a subset of the discontinuum D, as every countable set is homoeomorphic with a subset of D (Theorem of SIERPIŃSKI). Among all possible subsets of D which together form a family with potency $2^{\aleph}$ [2]), there are $\aleph$ (different) compact sets. So D too contains at most $\aleph$ compact countable subsets. The family of sets containing one point of D already gives $\aleph$ different compact countable sets. So D has exactly $\aleph$ compact countable subsets.

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[1]) A set of sets is called a family or a system.

[2]) $\aleph_0$ is the potency of the set of natural numbers, $\aleph$ that of the set of real numbers.

We now have to decide how many and which of these subsets are really *topologically different*. We shall find $\aleph_0$ topologically different compact countable sets and so of course just as many topological invariants.

1. 3. Let A be an arbitrary compact countable set. By taking the set I_0 of all isolated points out of A we have the set of all limit-points of A, the first derived set A_1, left. By taking out of A_1 the set of isolated points I_1 one has the second derived set A_2 left. This process is continued. It appears that all derived sets are compact. It is not possible that a certain (compact) derived set A_i is dense in itself; for every compact set, dense in itself, is homoeomorphic with the discontinuum D, contradictory to the countability of A. Only the following two possibilities may offer themselves: 1⁰. after a finite number of steps one finds a vacuous derived set A_n, or 2⁰. not one of the derived sets A_i (i being an arbitrary natural number) is vacuous. In the second case we determine the intersection of the compact sets $A = A_0, A_1, A_2, A_3, \ldots$ and thus achieve an apparently compact set $A_\omega = \prod_{i=0}^{\infty} A_i$, the ω-th derived set. Of A_ω we consider the derived set $A_{\omega+1}$ and this process is continued in the same way along the transfinite sequence of ordinal numbers. Always $A_\alpha = I_\alpha + A_{\alpha+1}$ for any ordinal number α, where I_α is the set of isolated points of A_α. This process will end after an at most countable number of steps (as A is countable), in other words, there is first ordinal number μ so that $A_\mu = 0$. Here we have a special case of the well-known theorem of CANTOR–BENDIXSON. As μ belongs to the first or second class of numbers, i.e. to the (well-ordered) set of all ordinal numbers fixing a countable potency, we may, supposing that ω_1 is the first ordinal number of the third class of numbers (therefore the first ordinal number of potency $\aleph_1$), denote the well-ordered family of derived sets by

$$A = A_0 \supset A_1 \supset A_2 \supset \ldots A_\omega \supset A_{\omega+1} \ldots A_{\omega.2} \supset \ldots A_\alpha \supset \ldots A_\mu = 0 \quad |\omega_1 \qquad (1)$$

A set A_β therefore is formed in one of the following ways: 1⁰. by removing the isolated points from $A_{\beta-1}$, 2⁰. if β has no immediately preceding ordinal number — if β is limes-number — by taking the intersection $\prod_{\gamma<\beta} A_\gamma$. We further remark that for every considered ordinal number α: $\bar{I}_\alpha = A_\alpha$, in other words the set of limit-points of I_α is exactly $A_{\alpha+1}$.

We may sharpen the theorem of CANTOR–BENDIXSON a little, as we are considering compact sets. Should μ be limes-number it would be possible to find a sequence of ordinal numbers $\gamma_1, \gamma_2, \gamma_3, \ldots$ with $\lim_{i\to\infty} \gamma_i = \mu$. This implies

$$\prod_i A_{\gamma_i} = A_\mu = 0.$$

On the other hand this intersection can not be vacuous as the A_{γ_i} are compact sets. Then μ is not a limes-number and has a predecessor. So

there is a last not-vacuous derived set A_λ ($\lambda + 1 = \mu$) and we may write (1) as

$$A = A_0 \supset A_1 \supset A_2 \ldots A_\omega \supset \ldots A_\alpha \supset \ldots A_\lambda \supset A_{\lambda+1} = 0 \quad | \omega_1. \quad (1')$$

A_λ then must be finite and consists of, say l points. So we have the following result: for every compact countable set A a "last" set A_λ may be found, consisting of l points. Thus to each A belongs a pair of numbers (λ, l), where λ is an ordinal number $< \omega_1$ and l a natural number. (λ, l) we call the *degree* of A. Conversely it is clear (with the assistance of a construction by means of transfinite induction) that for every pair of numbers (λ, l) of that kind there may be found a compact countable A, having (λ, l) for degree.

We now contend that those ordinal numbers (λ, l) give exactly all possible topologically different compact countable sets.

Theorem I. *Two compact countable sets A and B may be mapped topologically on each other only if the degree (λ, l) of A is the same as that of B. To each degree (λ, l) where λ is an arbitrary ordinal number of the first or second class of numbers and l is an arbitrary natural number, there belongs (topologically spoken) just one not-vacuous compact countable set A. Thus all topological invariants of the (family of) compact countable sets are known* [3]).

In 1. 4., 1. 5. and 1. 6. this theorem will be proved.

1. 4. It is desirable to give some lemmas and notions.

The degree (λ, l) is greater (by definition) than the degree (μ, m) if $\lambda > \mu$ or $\lambda = \mu$ and $l > m$. In the same way we define $=$ and $<$.

By the *degree of an* arbitrary *point* p in A we mean the highest ordinal number α for which p still belongs to A_α. Apparently there is indeed such a highest ordinal number α. It is evident that all points with degree α in A together form exactly the set I_α. If p has degree α and q has degree β the degree of p is greater than, equal to or smaller than that of q if $\alpha > \beta$, $\alpha = \beta$, $\alpha < \beta$ respectively. Apparently, if A has degree (λ, l) and p has degree α in A, then $\alpha \leqq \lambda$. Further the following simple lemmas hold true, which we shall mention without proofs.

Lemma I. The degree of a compact subset $D \subset A$ is smaller than or equal to that of A.

Lemma II. If a point $p \subset A$ has degree α there may be found a (compact) neighbourhood $U = U(p/A)$ of p in A so that the set U has degree $(\alpha, 1)$.

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[3]) The ordinal numbers (λ, l) are topological invariants. The property "to be of degree (1, 1) or of degree (2, 1)" also is a topological invariant (denoted by (1, 1) + (2, 1)). This topological invariant however follows trivially from the given system of invariants. It is easy to prove that indeed every topological invariant of the family in question is equivalent with a "sum of well-chosen numbers of degree".

Lemma III. If D is a *part-set* of A (i.e. an in A both open and closed set) then a point of D has the same degree in D as in A.

Lemma IV. If a point p has degree α in A, then in every neighbourhood of p there are points with degree $<\alpha$ but for the rest arbitrary. Further there may be found a sufficiently small neighbourhood $U(p)$ of p so that *all* points of that neighbourhood have a degree $\leqq \alpha$.

1. 5. In this number we shall prove that, if the compact countable sets A and B both have the degree (λ, l), there exists a topological mapping of A on B.

This is proved by means of transfinite induction to (λ, l) where we suppose all possible values of (λ, l) arranged according to size in a well-ordered sequence.

If A and B both have the degree $(0, l)$, in other words if A and B both consist of l points, then every one-to-one mapping of A on B is a topological mapping. Now let us consider the case where A and B have the degree $(\lambda, 1)$, while the theorem is proved for all compact countable sets with degree $<(\lambda, 1)$. Both in A and in B one may find one point, say a and b, with degree λ. Choose a system of neighbourhoods $U_1, U_2, \ldots$ of a consisting of part-sets with $U_i \supset U_{i+1}$. We put $R_1 = A - U_1$. The degree (ϱ_1, r_1) of R_1 is less than $(\lambda, 1)$. Using the lemmas mentioned above one easily sees that there may be found a part-set $V_1 = V_1(b)$ of b in B so that $S_1 = B - V_1$ has a degree $(\sigma_1, s_1) > (\varrho_1, r_1)$. Further we may certainly find in U_1 s_1 points with degree σ_1 and therefore also (sufficiently small) part-neighbourhoods of those s_1 points, each having the degree $(\sigma_1, 1)$. The sum of R_1 and of a well-chosen number of these s_1 part-neighbourhoods together form a part-set R_1' with degree (σ_1, s_1) [4]. According to the supposed induction R_1' may be mapped topologically on S_1.

We now proceed to the second step. Evidently there exists a sufficiently great value i so that U_i has no point in common with R_1'. Put

$$R_2 = A - U_i - R_1'.$$

Let the degree of this part-set R_2 be (ϱ_2, r_2). It is possible to find a part-neighbourhood $V_2 = V_2(b)$ within V_1 so that the degree of $S_2 = V_1 - V_2$ has a value $(\sigma_2, s_2) > (\varrho_2, r_2)$. Within U_i we may find at least s_2 points each with degree σ_2 and therefore just as many part-neighbourhoods of those points with degree $(\sigma_2, 1)$. R_2 and the sum of a number of those s_2 part-neighbourhoods form a set R_2' with degree (σ_2, s_2). We map R_2' topologically on S_2.

We now proceed to the third step. It is possible to find a sufficiently great $k > i$ so that U_k has no point in common with R_2'. Put

$$R_3 = U_i - U_k - U_i \cdot R_2'.$$

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[4]) If $\sigma_1 > \varrho_1$ we take all s_1 part-neighbourhoods. If $\sigma_1 = \varrho_1$ we take only $s_1 - r_1$ arbitrary part-neighbourhoods.

Construct a part-neighbourhood $V_3 = V_3\,(b)$ within V_2 so that $S_3 = V_2 - V_3$ has a degree $(\sigma_3, s_3) > (\varrho_3, r_3)$, where (ϱ_3, r_3) is the degree of R_3. In the above mentioned way we construct an R'_3 which is mapped topologically on S_3.

This process is continued ad infinitum, while we take care that the part-neighbourhoods $V_1, V_2, V_3, \ldots$ have b as intersection, in other words form a system of neighbourhoods of b. Then, when we map a on b this mapping together with the constructed mappings of R'_i on S_i gives a mapping of A on B which is evidently topological.

Finally let us suppose the case that A and B have the degree (λ, l) with $l > 1$ while the theorem has already been proved for the degree $(\lambda, 1)$. In A are exactly l points with degree λ. Determine for $l - 1$ of those points disjunct part-neighbourhoods $W_1, W_2, \ldots, W_{l-1}$ which moreover don't contain the last of the l points. To the l-th point we add the part-neighbourhood $A - \sum_{i=1}^{l-1} W_i$. Construct in the same way for B the part-sets $Z_1, Z_2, \ldots, Z_{l-1}, Z_l$. The part-sets W_i and Z_i apparently all have the degree $(\lambda, 1)$ and we may therefore map each W_i topologically on Z_i. These mappings together give a topological mapping of A on B [5]).

1. 6. Theorem I will be proved completely if we can show that a compact countable set A with degree (λ, l) may not be mapped topologically on a (compact countable) set B of different degree.

Let $t\,(A) = B$ be a topological mapping of A on B. Then the isolated points i_0 of A will necessarily be mapped on the isolated points k_0 of B: $t\,(I_0) = K_0$, so $t\,(A_1) = B_1$. In general $t\,(A_\alpha) = B_\alpha$ for a derived set A_α of A holds true. For, supposing this has already been proved for all ordinal numbers β with $\beta < \alpha$, we may prove it for α as follows. If α is not a limes-number then

$$t\,(A_{\alpha-1}) = B_{\alpha-1} \text{ and } t\,(I_{\alpha-1}) = K_{\alpha-1}, \text{ so } t\,(A_\alpha) = B_\alpha.$$

If α is a limes-number then

$$A_\alpha = \prod_{\beta<\alpha} A_\beta$$

and

$$t\,(A_\alpha) = t\,(\prod_{\beta<\alpha} A_\beta) = \prod_{\beta<\alpha} t\,(A_\beta) = \prod_{\beta<\alpha} B_\beta = B_\alpha.$$

From $t\,(A_\alpha) = B_\alpha$ follows in particular: $t\,(A_\lambda) = B_\lambda$, in other words B_λ is not vacuous and consists of exactly l points. In other words B has degree (λ, l), q.e.d.

5) Thanks to the kind criticism of HANS FREUDENTHAL I was able to shorten the proof of 1.5., which was at first far too long, considerably. He moreover sent me a somewhat shorter proof. By making use of some simple properties of the ordinal numbers one may prove (also by transfinite induction) that *every A of degree (λ, l) is homoeomorphic with the well-ordered set $\omega^\lambda . l + 1$.*

Remark. With a topological mapping of A on B the degree of a point is invariant.

1. 7. In this number let A be a not necessarily compact but only a microcompact countable set. We now consider in the transfinite sequence of the derived sets of the microcompact set A the first ordinal number μ for which A_μ is compact; A_μ is allowed to be vacuous. Let (λ, l) be the degree of A_μ. We define (μ, λ, l) as the degree of A. For a compact $A = A_0$ evidently $\mu = 0$ so that the above defined (λ, l) now is denoted by $(0, \lambda, l)$. If each not-vacuous A_α is not compact the degree of A is evidently $(\mu, 0, 0)$, while the degree of the vacuous set is $(0, 0, 0)$.

Now the following theorem holds true:

Theorem II. *The degrees* (μ, λ, l) $(0 \leqq \mu + \lambda < \omega_1;\ l = 0, 1, 2, \ldots;$ *if* $l = 0$ *then* $\lambda = 0)$ *determine precisely all topological invariants of the family of the microcompact countable sets, in other words a microcompact countable set may be mapped topologically only on a set having the same degree. In this way in particular all closed countable subsets of an n-dimensional Euclidian space have been classified.*

The Theorem will be proved in 1. 8.—1. 9.

1. 8. If the microcompact set A is not compact A may be compactified by one point P, in other words it is possible to find a set, homoeomorphic with A, which we also denote by A, and a point P so that $A + P$ is a compact countable set. Further this new space $A + P$ is completely determined by A in topological respect, in other words if we first compactify A by a point P to $A + P$ and afterwards a set A' homoeomorphic with A by a point P' to $A' + P'$, then $A + P$ may be mapped topologically on $A' + P'$, where P is mapped on P'. This follows from a simple theorem concerning topological extension (comp. J. DE GROOT, Proc. Kon. Ned. Ak. v. Wet. **44** (1941), p. 933): every topological mapping of a separable space A (i.e. a normal space with countable base [6])) on a separable space A' may be extended to a topological mapping of $A + P$ on $A' + P'$, if $\bar{A} = A + P$ and $\bar{A}' = A' + P'$ are compact separable spaces. Every (not compact) microcompact countable set A therefore completely determines the compact countable set $A + P$.

Concerning the degree (μ, λ, l) of A the following principally different possibilities may offer themselves:

1°. $l = 0$. The degree of A then is $(\mu, 0, 0)$ and that of $A + P$ necessarily $(0, \mu, 1)$.

2°. $l \neq 0$. The degree of A then is (μ, λ, l) $(l \neq 0)$ and that of $A + P$ necessarily $(0, \mu + \lambda, l)$ if $\lambda \neq 0$ or $(0, \mu + \lambda, l + 1)$ if $\lambda = 0$.

It is easy to see that for each degree (μ, λ, l) there is a microcompact A with (μ, λ, l) as degree.

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6) The theorem holds also true for more general, not necessarily metrical spaces.

1. 9. If A and B both have degree (μ, λ, l) then A may be mapped topologically on B. For, in case 1^0 $\bar{A} = A + P$ may be mapped topologically on the compactified $\bar{B} = B + Q$ as both have degree $(0, \mu, 1)$. According to the remark in 1. 6. P is mapped on Q, which implies that A is mapped topologically on B. If 2^0 is the case we divide $\bar{A}$ into the two disjunct part-sets $\bar{A}_1$ and $\bar{A}_2$ so that $\bar{A}_1$ contains point P and has degree $(0, \mu, 1)$ (the degree of P in $A + P$ is always μ), while $\bar{A}_2$ has degree $(0, \mu + \lambda, l)$. In the same way we may divide $\bar{B}$ into $\bar{B}_1$ and $\bar{B}_2$ where $\bar{B}_1$ will contain point Q. According to Theorem I we may map $\bar{A}_1$ topologically on $\bar{B}_1$ where P is mapped on Q and $\bar{A}_2$ (topologically) on $\bar{B}_2$. Then therefore A is mapped topologically on B.

Finally we have to prove that the topological image B of A has the same degree (μ, λ, l) as A. Let $t(A) = B$. We construct $\bar{A} = A + P$ and $\bar{B} = B + Q$. When we map P on Q this mapping together with t gives a topological mapping of $\bar{A}$ on $\bar{B}$. According to the remark in 1. 6. P and Q in $\bar{A}$ and $\bar{B}$ have the same degree while $\bar{A}$ and $\bar{B}$ have the same degree according to Theorem I. This immediately shows what we had to prove.

1. 10. The method developed above cannot be used in this shape to determine the topological classification for more general classes of countable sets, but by making a few modifications it is possible to obtain some still more general results. I did not, however, achieve much in this way and shall therefore refrain from mentioning the results. In this number we shall only show that the family of all countable sets, and even the family of the countable sets having a vacuous nucleus, dense in itself, contain $\aleph$ *topologically different* sets (and just as many so-called independent topological invariants). In an R_n (or in D) there apparently exist precisely $\aleph^{\aleph_0} = \aleph$ different countable subsets. Among these are, as we saw above $\aleph_0$ *topologically different microcompact* sets, but as we will show even $\aleph$ topologically different (not microcompact) countable sets.

Theorem III. *It is possible to construct* $\aleph$ *topologically different countable sets (all with a vacuous nucleus dense in itself). There exist exactly* $\aleph$ *topologically different countable sets.*

Proof. In 1. 3. we constructed the transfinite sequence of the derived sets of a compact countable set A. If A is not compact it is also possible to construct the sequence of the derived sets (every derived set being closed in A). In doing so however we may meet the complication that a certain derived set A_α (and even $A = A_0$ itself) is dense in itself, in other words does no longer contain any isolated points. Then the process is stopped. A_α then is called the nucleus, dense in itself, of A. If A does not contain a nucleus which is dense in itself we eventually find again a first vacuous derived set A_d (but μ now needs not necessarily have a predecessor).

Let us consider $A = I_0 + A_1$ while A has a vacuous nucleus. In a point

$a_1 \subset A_1$ the following two principally different possibilities may offer themselves: 1^0. a_1 has a (sufficiently small) compact neighbourhood $U(a_1/A)$: a_1 is microcompact in A; 2^0. a_1 is not microcompact in A, in other words within every neighbourhood $U(a_1/A)$ we may find a sequence of points of A without a limit-point in A. But then there even exists a sequence of points of I_0 without a limit-point (in A) within every mentioned $U(a_1/A)$ (for, the nucleus of A being vacuous, $\overline{I_0} = A$). In case 1^0. we denote a_1 by a_1^+, in case 2^0. by a_1^-. All points of I_0 are plus-points. The points of A_1 may contain both plus- and minus-points, where the set of minus-points is closed in A as is easily proved. We therefore may divide A_1 into $A_1^+ + A_1^-$. Let us now, without entering further into this general case, consider the following special case.

$I_1 = A_1 - A_2$ will consist exclusively of plus-points in $A_0 = A$. I_2 will, *with regard to the set* A_1, consist exclusively of minus-points, I_3 will, *with regard to the set* A_2, consist exclusively of plus-points. A_4 will be vacuous. Such a set we therefore may denote by $+-+$. In the same way we may construct sets $+--$, $-+-$, etc., where we may find all possible combinations. Let us now consider, however, a set where no A_i (i finite) is vacuous; then we may in the same way find all possible sets $\delta_1 \delta_2 \delta_3 \ldots\ldots$ ($\delta_i = +$ or $-$), for instance by constructing $\delta_1 \delta_2 \delta_3 \ldots\ldots$ as the sum of a countable number of sets δ_1, $\delta_1 \delta_2$, $\delta_1 \delta_2 \delta_3 \ldots\ldots$ etc. which are isolated in regard to each other. The potency of all possible different sets $\delta_1 \delta_2 \delta_3 \ldots$ is exactly $2^{\aleph_0} = \aleph$, therefore that of the continuum. What remains to be proved is that a set $A \equiv \delta_1 \delta_2 \delta_3 \ldots\ldots$ may not be mapped topologically on a (different) set $B \equiv \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots\ldots$. Suppose there existed a topological mapping $t(A) = B$. Then I_0 is necessarily mapped on K_0 (i.e. the set of isolated points of B), and so A_1 on B_1. Thus in general $t(A_\alpha) = B_\alpha$ (comp. 1. 6.; the proof given there holds also true for arbitrary countable sets) and so also $t(I_j) = K_j$. As $\delta_1 \delta_2 \delta_3 \ldots\ldots$ and $\varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots\ldots$ are unequal, there exists a first index j with $\delta_j \neq \varepsilon_j$. Let I_j consist of plus-points and K_j of minus-points with regard to A_{j-1} and B_{j-1}. As $t(A_{j-1}) = B_{j-1}$, a compact neighbourhood $U = U(i/A_{j-1})$ of a point $i \subset I_j$ in A_{j-1} is mapped on a compact neighbourhood $t(U) = V$ of $k = t(i)$ in B_{j-1} which contradicts the fact that k was a minus-point. Therefore there exist indeed $\aleph$ topologically different countable sets with vacuous nucleus.

II.

2. 1. In the following we shall determine all possible continuous invariants of the family of all countable sets. In other words we shall give a *continuous classification of the countable sets.* By a *continuous invariant* (with regard to a certain family F) is naturally meant a property which, if valid for a set A of F, also holds true for all continuous images $f(A)$ belonging to F. Trivial continuous invariants, i.e. properties valid for all

sets of F, will of course be left out of account. The property "to be countable" therefore is not a continuous invariant with regard to the family F of the countable sets, but the property "compactness" on the other hand is a continuous invariant in F. No continuous invariants for instance are the properties "to consist of infinitely many points", "to contain an isolated point", "to be dense in itself", etc. For a certain continuously invariant property α there are (in the family of the countable sets F) a number of sets with property α. We denote the family of those sets, which is a subset of F, by $E(\alpha)$. Therefore, if a set belongs to $E(\alpha)$, all its continuous images also belong to $E(\alpha)$. If $E(\beta) \subset E(\alpha)$ (i.e. every set of $E(\beta)$ is a set of $E(\alpha)$), property β evidently always implies property α. Notation $\beta \rightarrow \alpha$.

We consider in the first place an arbitrary not compact countable set and we shall show that this set may be mapped continuously on every countable set.

Theorem IV. *Every not compact countable set A may be mapped continuously on every countable set B.*

Proof. We determine a sequence belonging to A of different points $a_1, a_2, a_3, \ldots\ldots$, all isolated in respect to each other and having no limit-point belonging to A. As A is not compact it is evidently possible to construct such a sequence. Then we construct for every point a_i a neighbourhood $U_i = U(a_i)$ of a_i in A such that not a single pair of neighbourhoods has a point in common, while the diameter $[U_i]$ of U_i be less than $1/2i$. Moreover we see to it that every U_i is a part-set

B is countable and consists of the points $b_1, b_2, \ldots, b_k, \ldots$.

We define

$$f(U_i) = b_i \qquad f(A - \sum_{i=1}^{\infty} U_i) = b_1.$$

If B is finite and consists of k points we put $b_k = b_{k+1} = \ldots\ldots$ It is easy to see that f is continuous.

2. 2. We denote the property of compactness by $\varkappa$ and show that for an arbitrary continuous invariant $\alpha : \alpha \rightarrow \varkappa$. According to 2. 1. we have to prove $E(\alpha) \subset E(\varkappa)$. Suppose there existed a not compact set A having property α. According to Theorem IV then every countable set would have property α and then α would be a trivial continuous invariant, which was excluded. We have therefore achieved the required contradiction. We ascertain that if there exist continuous invariants this implies the property of compactness. From now on we therefore have to consider only the compact countable sets. We shall prove that these have only the following continuous invariants: "the degree is less than (λ, l)" (this implies compactness, as only for a compact countable set a degree (λ, l) has been defined). We denote this property by $[<\lambda, l]$. The continuous invariant "to be a finite set" for instance is determined by $[<1, 1]$. So we get the following continuous classification of the countable sets.

Theorem V. *In the topology of the countable sets there exist as the only continuous invariants the infinitely many properties of compactness* $\varkappa$ *and* $[<\lambda, l]$. *Between these invariants there apparently is the following connection: from* $[<\lambda, l]$ *always follows* $[<\mu, m]$ *and* $\varkappa$ *of* $(\mu, m) \leqq (\lambda, l)$.

The proof will be given in 2. 3. and 2. 4.

2. 3. In this number we prove

Theorem V'. *The compact countable set* A *may be mapped continuously on the (compact) countable set* B *only if the degree of* B *is less than or equal to that of* A.

Proof. Let the degree (α, m) of A be greater than or equal to the degree (β, n) of B. We have to map A continuously on B. This is done by means of transfinite induction to β. For $\beta = 0$, B consists of n points. As A consists of more than n points it will certainly be possible to divide A into n disjunct closed subsets $A_1, A_2, \ldots, A_n$. The points of each A_i are mapped on the i-th point of B. This is the required continuous mapping of A on B. Now let it be possible to construct a continuous mapping of A on B for all ordinal numbers $<\beta$. In B there are precisely n points $b_1, b_2, \ldots, b_n$ with degree β. The degree of all other points of B is $<\beta$. For each point $b_i (i = 1, 2, \ldots, n)$ we choose a sequence of (compact) part-neighbourhoods $U_i^j = U^j(b_i)$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots)$ of b_i in B so that:

$$U_i^j \supset U_i^{j+1}, \quad \prod_j U_i^j = b_i, \quad U_r^j \cdot U_s^j = 0 \qquad (r \neq s).$$

Further we put

$$S_0^0 = B - \sum_{i=1}^{n} U_i^1, \quad S_i^j = U_i^j - U_i^{j+1} \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots).$$

The sets S are evidently compact and have a degree $<(\beta, 1)$. We denote the degree of S_i^j $(i = 0, 1, \ldots, n;\ j = 0, 1, 2, \ldots)$ by (σ_i^j, s_i^j). As

$$(\alpha, m) > (\beta, n)$$

A certainly contains n points $a_1, a_2, \ldots, a_n$ of degree β. Using the lemmas of 1. 4. we may easily prove that it is possible to construct a (compact) part-set R_0^0 which does not contain a single point $a_i (i = 1, 2, \ldots, n)$ and the degree of which is $(\varrho_0^0, r_0^0) > (\sigma_0^0, s_0^0)$. The complement-set $A - R_0^0$ may be divided into n disjunct closed sets $V_i^1 (i = 1, 2, \ldots, n)$, where V_i^1 contains a_i precisely. For every V_i^1 we may construct a part-set $R_i^1 \subset V_i^1$ not containing a_i, while the degree (ϱ_i^1, r_i^1) of R_i^1 is greater than or equal to the degree (σ_i^1, s_i^1). After that we put $V_i^2 = V_i^1 - R_i^1$ and define in the same way an R_i^2 within V_i^2, etc. Because of the supposed induction it is now possible to map every $R_i^j (i = 0, 1, \ldots, n;\ j = 1, 2, \ldots)$ continuously on the corresponding S_i^j. Further, if we map a_i on b_i it is evident that in this way we have constructed a continuous mapping of A on B.

There remains to be proved: if the degree (α, m) of A is less than the degree (β, n) of B, then it is impossible to map A continuously on B. For $\alpha = 0$ A consists of m points and B of more than m points. Therefore a continuous mapping of A on B is impossible. Put $\alpha > 0$. Suppose there exists a continuous mapping f with $f(A) = B$. Consider the set

$$I_0 = A_0 - A_1$$

of the isolated points of A. The image $f(I_0)$ contains $K_0 = B_0 - B_1$, for suppose there existed an isolated point p of B not belonging to $f(I_0)$, then p also did not belong to $\overline{f(I_0)}$, but $f(A) = f(I_0) = \overline{f(I_0)} = B$, so p would not belong to B which is incorrect. Now consider a point $b_1 \subset B_1$. It is possible to find a sequence of points in K_0 converging to b_1. These points, as we have just shown, possess all inverse images belonging to I_0. Those inverse images have a limit-point $a_1 \subset A_1$. Now a_1 has to be mapped on b_1 by f. In this way each point of B_1 has an inverse image belonging to A_1, in other words $f(A_1) \supset B_1$. So there is a closed and therefore compact subset A_1^1 of A_1, mapped on $B_1 : f(A_1^1) = B_1$. Should A_1^1 be vacuous we should have reached a contradiction. If A_1^1 is not vacuous we continue as follows. Consider the set $A_1^1 - A_2^1$ (A_2^1 is the derived set of A_1^1) and apply to this and to B_1 the same method as we applied to $A_0 - A_1$ and B. Thus we find a compact set A_2^2 with $A_2^2 \subset A_2^1 \subset A_2$ and $f(A_2^2) = B_2$. This process is continued and thus the following sequence is formed:

$$A \supset A_1^1 \supset A_2^2 \supset \ldots\ldots.$$

The sets of this sequence are mapped by f on the sets of the sequence

$$B \supset B_1 \supset B_2 \supset \ldots\ldots.$$

If one of the sets A_i^i is vacuous we have already reached a contradiction. If none of the A_i^i is vacuous the intersection $\prod\limits_i A_i^i = A_\omega^\omega$ is compact and not vacuous and apparently $f(A_\omega^\omega) = B_\omega$, while $A_\omega^\omega \subset A_\omega$. With A_ω^ω and B_ω, as originally with A and B, the process is continued transfinitely. If we find a vacuous A_γ^γ with $\gamma < \alpha$ a contradiction will have been reached. If not one of the A_γ^γ is vacuous we find eventually (after a countable number of steps) a set A_α^α which is mapped on B_α (B_α is not vacuous because $\alpha \leqq \beta$). A_α^α as a subset of A_α consists of at most m points. B_α on the other hand consists of more than m points as $(\alpha, m) < (\beta, n)$, in contradiction to $f(A_\alpha^\alpha) = B_\alpha$.

2. 4. Suppose there existed besides the continuous invariants mentioned in Theorem V yet another continuous invariant β. We already proved $\beta \to \varkappa$ and so may confine ourselves again to the compact countable sets. Consider $E(\beta)$; if this contains a set with degree (λ, l) it also contains all sets with degree $\leq (\lambda, l)$, according to Theorem V′. We order the

degree-numbers (λ, l) according to size in a well-ordered set X. Now two possibilities may offer themselves: 1^0. in X there is a first element (μ, m) so that the sets with degree (μ, m) don't belong to $E(\beta)$; 2^0. such an element (μ, m) may not be found in X, in other words $E(\beta)$ contains exactly all compact countable sets (for every compact countable set has a certain degree), or $\beta \equiv \varkappa$. In the first case all sets with degree $\geqq (\mu, m)$ necessarily possess property $-\beta$ (i.e. β does not hold true), for suppose there existed a set with degree $(\nu, n) \geqq (\mu, m)$ having property β, then every set with degree (μ, m) also would possess property β. Thus it appears that a compact countable set has property β only if its degree is $< (\mu, m)$, but that means: $\beta \equiv [< \mu, m]$.

As the properties $[< \lambda, l]$ and "compactness" according to Theorem V′ are continuous invariants and according to the above there exist no other continuous invariants, Theorem V has thus been proved.

The Hague, April 1945.

Mathematics. — *Sur la théorie métrique des approximations diophantiques.* - By J. F. KOKSMA. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 27, 1945.)

§ 1. *Introduction.*

1. On doit à M. A. KHINTCHINE [1]) le théorème suivant bien connu

Théorème 1. *Si $\varphi(x)$ désigne une fonction positive, monotone, non croissante du nombre naturel x, alors presque partout* [2]) *sur l'axe réel (des nombres θ) l'inégalité*

$$|\theta x - y| < \varphi(x) \quad \ldots \ldots \ldots \quad (1)$$

n'admet qu'un nombre fini de solutions entières $x \geq 1$, y, si la série

$$\sum_{x=1}^{\infty} \varphi(x) \quad \ldots \ldots \ldots \quad (2)$$

converge et possède une infinité de telles solutions, si la série (2) diverge.

La démonstration de la première partie (cas de convergence) est bien facile [3]). La deuxième partie (cas de divergence), M. KHINTCHINE la démontra de deux manières différentes, une fois se basant sur la théorie des fractions continues et plus tard sans faire usage de cette théorie [4]). La deuxième méthode lui offra la possibilité de considérer au lieu d'une seule inégalité (1) le système des $n \geq 1$ inégalités simultanées

$$|\theta_\nu x - y_\nu| < (\varphi(x))^{1/n} \qquad (\nu = 1, 2, \ldots, n).$$

Comme j'ai démontré antérieurement [5]), utilisant l'idée fondamentale de la démonstration de M. KHINTCHINE, il est possible de remplacer ces inégalités dans le théorème de M. KHINTCHINE par les inégalités plus générales

$$|\theta_\nu f_\nu(x) - y_\nu| < \varphi_\nu(x) \qquad (\nu = 1, 2, \ldots n)$$

pour une catégorie étendue de systèmes de $n \geq 1$ suites

$$f_\nu(1), f_\nu(2), \ldots \qquad (\nu = 1, 2, \ldots, n)$$

en remplaçant la série (2) par la série

$$\sum_{x=1}^{\infty} \varphi_1(x)\, \varphi_2(x) \ldots \varphi_n(x).$$

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[1]) A. KHINTCHINE, Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. Math. Ann. **92**, 115—125 (1924).

A. KHINTCHINE, Zur metrischen Theorie der Diophantischen Approximationen, Math. Z. **24**, 706—714 (1926).

[2]) Dans le sens de LEBESGUE.

[3]) Elle se trouve dans la première communication 1).

[4]) Resp. dans la première et dans la deuxième communication 1).

[5]) J. F. KOKSMA, Contribution à la théorie métrique des approximations diophantiques non-linéaires. Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 176—183, 263—268 (1942).

2. Dans le présent mémoire je développerai un théorème général (le théorème 4), analogue au théorème 1 (où dans l'inégalité (1) l'expression $\theta x - y$ sera remplacée par l'expression plus générale $F(x, \theta) - y - \alpha(x)$), mais ne contenant pas celui ci comme cas spécial. Cependant ce théorème permet des applications à des classes bien étendues de fonctions transcendantes $F(x, \theta)$, par exemple à la fonction exponentielle θ^x, comme le montre le théorème 2 qui est contenu dans le théorème 4 comme cas spécial [6]).

Théorème 2. *Soit $f(x)$ une fonction positive du nombre naturel x, telle que $f(x+1) - f(x) \geqq \sigma$, où σ désigne un nombre positif, indépendant de x. Soit $\varkappa(x)$ une fonction positive du nombre naturel x, telle que $\varkappa(x) f(x)$ soit une fonction monotone, non décroissante par rapport à $x \geqq \bar{x}_0$. Soit enfin α un nombre réel donné et $\varphi(x)$ une fonction positive, monotone, non croissante du nombre naturel x.*

Alors presque partout sur $\theta \geqq 1$ l'inégalité

$$-\varphi(x) \leqq \varkappa(x)\, \theta^{f(x)} - y - \alpha \leqq \varphi(x)$$

n'admet qu'un nombre fini de solutions entières $x \geqq 1$, y, si la série (2) converge, tandis que l'inégalité

$$0 < \varkappa(x)\, \theta^{f(x)} - y - \alpha < \varphi(x)$$

de même que l'inégalité

$$-\varphi(x) < \varkappa(x)\, \theta^{f(x)} - y - \alpha < 0$$

possède une infinité de telles solutions, si (2) diverge.

Remarques. 1. Il est sousentendu que $f(x)$, $\varkappa(x)$, $\varphi(x)$, $\bar{x}_0$, σ et α ne dépendent pas de θ. De même dans ce qui suit les grandeurs considérées ne dépendront pas du paramètre θ, si cela n'est pas exprimé explicitement dans le texte ou par la notation.

2. Les conditions du théorème 2 seront remplies, si l'on y pose

$$\varkappa(x) = 1,\ f(x) = x,\ \bar{x}_0 = 1,\ \sigma = 1 \qquad \text{et donc } \varkappa(x)\, \theta^{f(x)} = \theta^x.$$

De même le théorème 4 contient comme cas spécial [7]) le

Théorème 3. *Soit $f(x)$ une fonction positive et croissante du nombre naturel x, telle que pour tout couple de nombres entiers $x \geqq 1$, $k \geqq 1$*

$$\sum_{i=0}^{k-1} \frac{f(x+i)}{f(x+k)} \leqq C \qquad (C \text{ ne dépendant pas de } x, k). \quad . \quad . \quad (3)$$

Soit α un nombre réel et $\varphi(x)$ une fonction positive, monotone, non croissante du nombre naturel x. Alors presque partout sur l'axe réel des nombres θ, l'inégalité

$$-\varphi(x) \leqq \theta f(x) - y - \alpha \leqq \varphi(x)$$

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[6]) Pour la démonstration, voir le § 2.

[7]) Pour la démonstration, voir le § 3.

n'admet qu'un nombre fini de solutions entières $x \geqq 1$, y si la série (2) *converge, tandis que l'inégalité*

$$0 < \theta f(x) - y - \alpha < \varphi(x),$$

de même que l'inégalité

$$-\varphi(x) < \theta f(x) - y - \alpha < 0$$

possède une infinité de telles solutions, si (2) *diverge.*

Remarque. La condition (3) sera remplie, si $f(x+1) \geqq \tau f(x)$, où $\tau > 1$ ne dépend pas de x.

3. Formulons maintenant le

Théorème 4A. Conditions. I. *Soient a et $b (a < b)$ des nombres réels donnés, x_0 un nombre naturel donné et $F(x, \theta)$ pour chaque nombre θ du segment $a \leqq \theta \leqq b$ une fonction positive du nombre naturel $x \geqq x_0$, telle que*

$$F(x, \theta) \to \infty, \quad \text{si } x \to \infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

et en outre pour chaque nombre naturel $x \geqq x_0$ une fonction dérivable par rapport à θ, dont la dérivée $F'_\theta(x, \theta)$ soit positive, monotone, non décroissante, par rapport à θ et pour chaque θ du segment $a \leqq \theta \leqq b$ soit monotone, non décroissante par rapport à $x \geqq x_0$, de sorte que

$$F'_\theta(x, \theta) \to \infty, \text{ si } x \to \infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Enfin soient $\alpha(x)$ et $\beta(x)$ deux fonctions réelles du nombre naturel x, telles que

$$\varphi(x) = \beta(x) - \alpha(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

soit une fonction positive, monotone, non croissante de x.

II. *Supposons que la série* (2) *converge.*

Assertion. *Alors presque partout sur le segment $a \leqq \theta \leqq b$ l'inégalité*

$$\alpha(x) \leqq F(x, \theta) - y \leqq \beta(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

n'admet qu'un nombre fini de solutions entières $x \geqq 1$, y.

Théorème 4B. Conditions. I. *Supposons les conditions I du théorème 4A remplies.*

II. *Supposons que la série* (2) *diverge. Supposons enfin que pour chaque couple de nombres entiers $x \geqq x_0$, $k \geqq 1$, on ait*

$$\sum_{i=0}^{k-1} \frac{F'_\theta(x+i, \theta)}{F'_\theta(x+k, \theta)} \leqq C,$$

C désignant une constante positive, indépendante de x, k, et θ.

Assertion. *Alors presque partout sur le segment $a \leqq \theta \leqq b$ l'inégalité*

$$\alpha(x) < F(x, \theta) - y < \beta(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

possède une infinité de solutions entières $x \geqq 1$, y.

Le lecteur peut trouver la preuve du théorème 4A dans le § 8.

4. Comme nous démontrerons dans le § 4, le théorème 4B est contenu dans le théorème 4 B′ suivant, qui concerne une classe de fonctions $F(x, \theta)$ un peu plus étendue que celle du théorème 4B, mais par contre contient quelques restrictions par rapport à la fonction $\varphi(x)$.

Théorème 4B′. I. *Supposons les conditions I du théorème 4A remplies.*

II. *Soit le nombre naturel $\Lambda(x)$ une fonction monotone, non décroissante du nombre naturel x, telle que $\Lambda(x) \to \infty$, si $x \to \infty$ et telle que*

$$\sum_{x=1}^{\infty} \varphi(\Lambda(x))$$

diverge; soient γ et δ deux nombres positifs, indépendants de x tels que

$$2\gamma + \delta < 1 \quad \dots\dots\dots\dots \quad (9)$$

et $K = K(x)$ l'indice $\geqq 0$ tel que [8])

$$\sum_{k=0}^{K-1} \varphi(\Lambda(x+k)) \leqq \gamma < \sum_{k=0}^{K} \varphi(\Lambda(x+k)) \; (x \geqq x_0) \quad \dots\dots \quad (10)$$

Enfin supposons que pour chaque entier $x \geqq x_0$ tel que $K(x) \geqq 1$, chaque entier k sur $1 \leqq k \leqq K(x)$ et chaque nombre θ sur $a \leqq \theta \leqq b$, on ait

$$\sum_{i=0}^{k-1} \frac{F'_\theta(\Lambda(x+i), \theta)}{F'_\theta(\Lambda(x+k), \theta)} \leqq \delta. \quad \dots\dots\dots\dots \quad (11)$$

Alors l'assertion du théorème 4B est juste.

Si l'on pose dans ce théorème $\Lambda(x) = x$, on voit d'un coup d'oeil qu'il entraine le

Théorème 4B″. I. *Supposons les conditions I du théorème 4A remplies.*

II. *Soit la série (2) divergente; soient γ et δ deux nombres positifs, indépendants de x, tels que*

$$2\gamma + \delta < 1 \quad \dots\dots\dots\dots \quad (12)$$

et soit $K = K(x)$ l'indice $\geqq 0$ tel que [8])

$$\sum_{k=0}^{K-1} \varphi(x+k) \leqq \gamma < \sum_{k=0}^{K} \varphi(x+k) \quad (x \geqq x_0); \quad \dots\dots \quad (13)$$

supposons en outre que, si $K(x) \geqq 1$, pour chaque entier k sur $1 \leqq k \leqq K(x)$ et tout nombre θ sur $a \leqq \theta \leqq b$

$$\sum_{i=0}^{k-1} \frac{F'_\theta(x+i, \theta)}{F'_\theta(x+k, \theta,} \leqq \delta, \text{ si } x \geqq x_0. \quad \dots\dots \quad (14)$$

Alors l'assertion du théorème 4B est vraie.

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[8]) Pour $K = 0$, nous posons $\sum_{k=0}^{K-1} = 0$.

Remarque. Comme nous démontrerons aisément dans le § 5, non seulement le théorème 4B′ contient le théorème 4B″, mais à son tour le théorème 4B″ contient le théorème 4B′: c'est à dire que les deux théorèmes sont équivalents; nous pouvons donc nous restreindre à la démonstration du théorème 4B″, qu'on peut trouver dans le § 9.

5. Comme application du théorème 4B′ je démontre dans le § 6 le

Théorème 5. *Soit* ω *un nombre positif* $\leqq 1$, *soit* α *un nombre réel et soit* $\varphi(x)$ *une fonction positive, monotone, non croissante, telle que la série*

$$\sum_{x=1}^{\infty} \varphi\left(\left[x^{1/\omega}\right]\right)$$

diverge. Alors presque partout sur la demie-droite $\theta \geqq 1$ *l'inégalité*

$$\alpha < \theta^{x^{\omega}} - y < \alpha + \varphi(x)$$

de même que l'inégalité

$$\alpha - \varphi(x) < \theta^{x^{\omega}} - y < \alpha$$

possède une infinité de solutions entières $x \geqq 1$, y.

Remarque. Par exemple on peut poser $\varphi(x) = \dfrac{1}{x^{\omega} \log x}$.

Dans ce mémoire $[u]$ désigne l'entier de u.

6. Dans une communication antérieure, j'ai démontré par une toute autre méthode que pour une certaine catégorie de fonctions $F(x, \theta)$ $(a \leqq \theta \leqq b;\ x = 1, 2, \dots)$ presque partout sur le segment considéré $a \leqq \theta \leqq b$ l'inégalité

$$\alpha < F(x, \theta) - y < \alpha + \frac{\psi(x)}{x^{1/3}}$$

de même que l'inégalité

$$\alpha - \frac{\psi(x)}{x^{1/3}} < F(x, \theta) - y < \alpha$$

possède une infinité de solutions entières $x \geqq 1$, y, si $\psi(x)$ désigne une fonction croissante quelconque du nombre naturel x et α un nombre réel quelconque [9]).

Entre ce résultat et les théorèmes du présent article il y a une différence essentielle par rapport au choix du nombre α: dans le présent mémoire l'ensemble de mesure nulle dépend du choix de α et dans la communication citée on peut choisir encore arbitrairement le nombre α après avoir fixé le nombre θ. Mais abstraction faite de ce détail il est clair que pour toute fonction $F(x, \theta)$ où le théorème 4B est applicable, ce théorème présente une amélioration considérable. Quant au théorème 4B′, il y a des cas où le

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[9]) J. F. KOKSMA, Metrisches zur Theorie der Diophantischen Approximationen. Proc. Kon. Akad. v. Wetensch., Amsterdam, 39, 225—240 (1936).

résultat fourni est meilleur que le résultat antérieur cité, mais de même il y a des cas où le théorème 4B′ n'amène aucune amélioration. Par exemple, le théorème antérieur vaut dans le cas où $F(x, \theta) = x^{\theta}$ $(\theta \geqq 1,$ $x = 1, 2, \ldots)$, mais pour cette fonction le théorème 4B′ donne un résultat fort mauvais. Aussi dans le cas où $F(x, \theta) = \theta^{x^{\omega}}$ $(\theta \geqq 1, x = 1, 2, \ldots)$ considéré dans le théorème 5, ce théorème apporte une amélioration des résultats connus, si $\omega > \frac{1}{3}$, mais non pas, si $\omega < \frac{1}{3}$.

7. Dans cet article je me suis borné à considérer une seule inégalité (8), bien que la méthode employée permet de considérer des inégalités simultanées de cette forme.

§ 2. *Démonstration que le théorème 2 est contenu dans les théorèmes 4A, B.*

Appliquons le théorème 4A en posant

$$1 < a < b, \quad x_0 \geqq \operatorname{Max}\left(\bar{x}_0, \frac{1+\sigma}{\sigma}\right)$$

$$F(x, \theta) = \varkappa(x)\,\theta^{f(x)}, \text{ donc } F'_{\theta}(x, \theta) = \varkappa(x) f(x)\,\theta^{f(x)-1},$$

$$\alpha(x) = a - \varphi(x), \quad \beta(x) = a + \varphi(x)$$

et avec $2\varphi(x)$ au lieu de $\varphi(x)$. Alors la première partie du théorème 2 (cas de convergence) découle immédiatement du théorème 4A.

Pour démontrer la deuxième partie, nous posons

$$\alpha(x) = a, \ \beta(x) = a + \varphi(x), \text{ resp. } \alpha(x) = a - \varphi(x), \ \beta(x) = a.$$

Remarquons que, lorsque $x \geqq x_0$

$$\sum_{i=0}^{k-1} \frac{F'_{\theta}(x+i, \theta)}{F'_{\theta}(x+k, \theta)} = \sum_{i=0}^{k-1} \frac{\varkappa(x+i) f(x+i)}{\varkappa(x+k) f(x+k)}\, \theta^{-(f(x+k)-f(x+i))}$$

$$\leqq \sum_{i=0}^{k-1} a^{-\sigma(k-i)} = \sum_{j=1}^{k} a^{-\sigma j} < \frac{1}{a^{\sigma}-1},$$

c'est à dire que les conditions du théorème 4B sont remplies avec $C = \dfrac{1}{a^{\sigma}-1}$ pour tout segment $a \leqq \theta \leqq b$ où $1 < a < b$. D'où le théorème 2.

§ 3. *Démonstration que le théorème 3 est contenu dans les théorèmes 4A, B.*

Il saute aux yeux que nous pouvons nous restreindre au cas où $\theta > 0$, parce que le cas où $\theta < 0$ se ramène facilement à celui ci par changement de signe dans les inégalités considérées. Appliquons le théorème 4A en posant $0 < a < b$,

$$F(x, \theta) = \theta f(x), \text{ donc } F'_{\theta}(x, \theta) = f(x),$$

$$\alpha(x) = a - \varphi(x), \quad \beta(x) = a + \varphi(x)$$

et avec $2\varphi(x)$ au lieu de $\varphi(x)$. Alors la première partie du théorème 3 résulte immédiatement du théorème 4A.

Pour démontrer la deuxième partie, nous posons

$$\alpha(x)=a,\ \beta(x)=a+\varphi(x),\ \text{resp.}\ \alpha(x)=a-\varphi(x),\ \beta(x)=a.$$

Remarquons que

$$\sum_{i=0}^{k-1}\frac{F'_\theta(x+i,\theta)}{F'_\theta(x+k,\theta)}=\sum_{i=0}^{k-1}\frac{f(x+i)}{f(x+k)}\leqq C$$

(d'après (3)), c'est à dire que les conditions du théorème 4B sont remplies. La deuxième assertion du théorème 3 en est une conséquence immédiate.

§ 4. *Démonstration que le théorème* 4B *est contenu dans le théorème* 4B′.

Posons dans le théorème 4B′:

$\Lambda(x)=Nx$, où N est un nombre entier $>2C$, C désignant la constante dans le théorème 4B et en outre $\gamma=\frac{1}{6}$, $\delta=\frac{1}{2}$.

Alors $\varphi(x)$ étant une fonction monotone, non croissante, nous aurons

$$\sum_{x=1}^{M}\varphi(Nx)\geqq\frac{1}{N}\sum_{x=N}^{MN+N-1}\varphi(x)\to\infty,\ \text{si}\ M\to\infty,$$

parce que (2) diverge. Ça veut dire que de même la série

$$\sum_{x=1}^{\infty}\varphi(Nx)$$

diverge. Alors $K(x)$ étant désigné par (10), le membre gauche de (11) pour $1\leqq k\leqq K(x)$ et $x\geqq x_0$ est égal à

$$\sum_{i=0}^{k-1}\frac{F'_\theta((x+i)N,\theta)}{F'_\theta((x+k)N,\theta)}\leqq\frac{1}{N}\sum_{j=0}^{Nk-1}\frac{F'_\theta(Nx+j,\theta)}{F'_\theta(Nx+Nk,\theta)}\leqq\frac{C}{N}<\tfrac{1}{2}=\delta$$

d'après les conditions du théorème 4B. Toutes les conditions du théorème 4B′ étant remplies, il en découle que l'assertion du théorème 4B est juste.

§ 5. *Démonstration que les théorèmes* 4B′ *et* 4B″ *sont équivalents.*

1. Comme nous avons remarqué dans le § 1, le théorème 4B″ résulte du théorème 4B′ immédiatement.

2. Démontrons que le théorème 4B″ entraine le théorème 4B′. A cet effet nous posons

$$F^*(x,\theta)=F(\Lambda(x),\theta),\quad \alpha^*(x)=\alpha(\Lambda(x)),$$
$$\beta^*(x)=\beta(\Lambda(x)),\quad \varphi^*(x)=\varphi(\Lambda(x)),$$

où $F,\alpha,\beta,\varphi,\Lambda$ désignent les fonctions du théorème 4B′. Alors les conditions I du théorème 4A seront remplies, si l'on y remplace F,α,β,φ par $F^*,\alpha^*,\beta^*,\varphi^*$. En outre les conditions II du théorème 4B′ entrainent que

$K(x)$ suffit à la condition (13), si l'on y remplace la fonction $\varphi(x)$ par $\varphi^*(x)$. Envisageons maintenant l'expression

$$\sum_{i=0}^{k-1} \frac{F_\theta^{*\prime}(x+i,\theta)}{F_\theta^{*\prime}(x+k,\theta)} = \sum_{i=0}^{k-1} \frac{F'_\theta(\Lambda(x+i),\theta)}{F'_\theta(\Lambda(x+k),\theta)} \leqq \delta$$

d'après (11), si $1 \leqq k \leqq K(x)$. C'est à dire que (14) sera rempli, si l'on y remplace F par F^*.

Ainsi, toutes les conditions du théorème 4B'' étant remplies, ce théorème nous apprend que presque partout sur $a \leqq \theta \leqq b$, l'inégalité

$$\alpha^*(x) < F^*(x, a) - y < \beta^*(x),$$

c'est à dire

$$\alpha(\Lambda(x)) < F(\Lambda(x), \theta) - y < \beta(\Lambda(x))$$

possède une infinité de solutions entières $x \geqq 1$, y, mais cela veut dire que presque partout sur $a \leqq \theta \leqq b$ l'inégalité (8) possède une infinité de telles solutions.

§ 6. *Démonstration que le théorème 5 est contenu dans le théorème 4B'.*

Appliquons le théorème 4B' en posant $1 < a < b$, $x_0 \geqq 1$,

$$F(x,\theta) = \theta^{x^\omega}, \text{ donc } F'_\theta(x,\theta) = x^\omega \theta^{x^\omega - 1},$$

$$\alpha(x) = \alpha,\ \beta(x) = \beta + \varphi(x), \text{ resp. } \alpha(x) = \alpha - \varphi(x),\ \beta(x) = \alpha.$$

Alors les conditions I du théorème 4A sont remplies. Quant aux conditions II du théorème 4B', nous posons

$$\Lambda(x) = [(Nx)^{\frac{1}{\omega}}],\ \gamma = \tfrac{1}{6},\ \sigma = \tfrac{1}{2},$$

où N désigne un nombre naturel fixe, tel que $\frac{a}{a^N - 1} < \frac{1}{2}$. Alors d'après les conditions du théorème 5 la série

$$\sum_{x=1}^{\infty} \varphi(\Lambda(x)) = \sum_{x=1}^{\infty} \varphi([(Nx)^{\frac{1}{\omega}}])$$

diverge[10]. Si $K(x)$ désigne l'indice défini par (10), nous aurons pour $x \geqq x_0$ et $1 \leqq k \leqq K(x)$

$$\sum_{i=0}^{k-1} \frac{F'_\theta(\Lambda(x+i),\theta)}{F'_\theta(\Lambda(x+k),\theta)} \leqq \sum_{i=0}^{k-1} a^{[(N(x+i))^{\frac{1}{\omega}}]^\omega - [(N(x+k))^{\frac{1}{\omega}}]^\omega} \leqq \sum_{i=0}^{k-1} a^{N(x+i) - N(x+k) + 1}$$

parce qu'on a pour $u > 1$

$$(u^{\frac{1}{\omega}} - 1)^\omega = (u-1)\,\omega\,(v^{\frac{1}{\omega}} - 1)^{\omega - 1} \tfrac{1}{\omega} v^{\frac{1}{\omega} - 1} = (u-1)\left(\frac{v^{\frac{1}{\omega}} - 1}{v^{\frac{1}{\omega}}}\right)^{\omega - 1}$$

$$> u - 1 \qquad (1 < v < u).$$

---

10) Cf. le raisonnement analogue dans le § 4.

Alors on a

$$\sum_{i=0}^{k-1} \frac{F'_\theta(\Lambda(x+i),\theta)}{F'_\theta(\Lambda(x+k),\theta)} \leqq a \sum_{j=1}^{k} a^{-Nj} < \frac{a}{a^N - 1} < \tfrac{1}{2} = \delta.$$

Toutes les conditions du théorème 4B′ étant remplies, le théorème 5 en résulte immédiatement.

§ 7. *Quelques remarques sur les théorèmes* 4A *et* B″.

I. Sans nuire à la généralité nous pouvons supposer

$$0 \leqq \alpha(x) < 1 \quad \text{et} \quad \varphi(x) < 1, \text{ si } x \geqq x_0$$

et donc

$$\beta(x) = \alpha(x) + \varphi(x) < 2 \quad \text{si } x \geqq x_0.$$

II. Désignons par $\theta = \Phi(x, y)$ la fonction inverse de $y = F(x, \theta)$ par rapport à $\theta (a \leq \theta \leq b)$. Il est clair que $\Phi(x, y)$ est une fonction croissante par rapport à y sur le segment $F(x, a) \leq y \leq F(x, b)$ et que $\Phi'_y(x, y)$ est monotone, non croissante sur ce segment, le nombre $x > x_0$ étant supposé fixe.

Remarquons finalement que sur $a \leq x \leq b$

$$\frac{1}{F'_\theta(x, \theta)} \leqq \frac{1}{F'_\theta(x, a)} \to 0, \text{ si } x \to \infty,$$

c'est à dire que

$$\Phi'_y(x, y) = \frac{1}{F'_\theta(x, \Phi(x, y))} \to 0, \text{ si } x \to \infty$$

uniformément en y sur le segment

$T(x) \ldots \qquad\qquad F(x, a) \leqq y \leqq F(x, b).$

III. On voit d'un coup d'oeil qu'il suffira de démontrer les théorèmes 4 presque partout sur $a' \leq \theta \leq b'$, lorsque a', b' désignent deux nombres arbitraires, pour lesquels $a < a' < b' < b$. Supposons ces nombres fixés durant le reste de ce mémoire. Alors nous pouvons supposer

$$x_1 = x_1(a', b') \geq x_0,$$

assez grand pour qu'on ait pour $x \geq x_1$

$$\Phi'_y(x, y) < Min\,(b - b', a' - a) \text{ sur } T(x)$$

et en outre

$$F(x, a) + 2 < F(x, a') < F(x, a') + 2 < F(x, b') - 2$$
$$< F(x, b') < F(x, b) - 2,$$

car pour $a \leq a'' < b'' \leq b$ on a

$$F(x, b'') - F(x, a'') \geqq (b'' - a'')\, F_\theta(x, a) \to \infty, \text{ si } x \to \infty.$$

Les nombres a' et b' étant fixés, nous pouvons choisir deux nombres arbitraires A et B pour lesquels $a' < A < B < b'$ et supposer

$$x_2 = x_2 (A, B) \geqq x_1$$

assez grand, pour que de plus

$$F(x, A) + 2 < F(x, B) - 2, \text{ si } x \geqq x_2.$$

IV. Dans tous les lemmes suivants (1—8) les conditions I du théorème 4A sont sousentendues. Les conditions II du théorème 4B″ ne rentreront que dans le lemme 8.

§ 8. *Démonstration du théorème* 4A.

1. Démontrons d'abord le

Lemme 1. *Si les conditions* I *du théorème* 4A *sont remplies et* a', b', x_1 *désignent les nombres de la* III-*ième remarque du* § 7, *alors afin que les nombres entiers* $x \geqq x_1$, y *et le nombre réel* θ $(a' \leqq \theta \leqq b')$ *satisfassent à l'inégalité* (7), *il faut que* y *appartient au segment*

$$F(x, a') - \beta(x) \leqq y \leqq F(x, b') - \alpha(x) \quad . \quad . \quad . \quad . \quad (15)$$

et que θ *appartient au segment*

$$P(x, y) \ldots . - \varphi(x)\, \Phi'_y(x, y + \alpha(x)) \leqq \theta - \Phi(x, y + \beta(x)) \leqq 0. \quad (16)$$

Démonstration. De (7) et (6) découle

$$F(x, \theta) = y + \beta(x) - \vartheta\, \varphi(x) \quad (0 \leqq \vartheta \leqq 1),$$

d'où (15) et d'où en outre

$$\begin{aligned} \theta = \Phi(x, y + \beta(x) - \vartheta\, \varphi(x)) &= \Phi(x, y + \beta(x)) \\ &- \vartheta\, \varphi(x)\, \Phi'_y(x, y + \beta(x) - \vartheta'\, \varphi(x)) \end{aligned}$$

$(0 \leqq \vartheta' \leqq \vartheta)$, d'où (16).

2. Soit $x \geqq x_1$ un nombre entier, θ un nombre sur $a' \leqq \theta \leqq b'$. Afin que (7) soit rempli, il faut d'après le lemme 1 que (15) et (16) soient remplis.

Désignons par $E'(x)$ l'ensemble des nombres θ du segment $a' \leqq \theta \leqq b'$ auxquels au moins un nombre y correspond tel que (7) soit rempli. Alors on a donc

$$m\, E'(x) \leqq \Sigma\, \varphi(x)\, \Phi'_y(x, y + \alpha(x))$$

où le signe de sommation s'étend sur les entiers y appartenants au segment (15), c'est à dire que

$$\begin{aligned} m\, E'(x) &\leqq \varphi(x) \left\{ \int\limits_{F(x, a') - \alpha(x)}^{F(x, b') - \alpha(x)} \Phi'_y(x, y + \alpha(x))\, dy + 2\, \Phi'_y(x, F(x, a') - \varphi(x)) \right\} \\ &\leqq \varphi(x) \{ b' - a' + 2\, \Phi'_y(x, F(x, a') - 1) \} \\ &\leqq 2\, \varphi(x)(b' - a'), \text{ si } x \geqq x_1^*(a', b') \geqq x_1. \end{aligned}$$

Considérons maintenant l'ensemble E' des nombres θ du segment $a' \leqq \theta \leqq b'$ pour lesquels (7) possède une infinité de solutions entières $x \geqq 1$, y. Alors θ est contenu dans tous les ensembles

$$E^*(x) = \sum_{k=0}^{\infty} E'(x+k) \quad (x \geqq 1),$$

donc dans l'ensemble produit E^* des ensembles

$$E^*(1) \supset E^*(2) \supset \ldots .$$

Or, alors l'ensemble E' pour tout nombre naturel $x \geqq x_1^*$ possède la mesure

$$m\,E' \leqq m\,E^* \leqq m\,E^*(x) \leqq \sum_{k=0}^{\infty} m\,E'(x+k)$$
$$\leqq 2\,(b'-a') \sum_{k=0}^{\infty} \varphi(x+k) < \varepsilon$$

pour tout nombre positif ε, si $x \geqq x_2^* = x_2^*(\varepsilon) \geqq x_1^*$, en vertu de la convergence de la série (2). C'est à dire que $mE' = 0$. C.f.d.

§ 9. *Démonstration du théorème* 4B″.

1. Démontrons d'abord quelques lemmes.

Lemme 2. *Si les conditions* I *du théorème* 4A *sont remplies, et* a', b', x_1 *désignent les nombres de la* III-*ième remarque du* § 7, *alors afin que les nombres entiers* $x \geqq x_1$, y *et le nombre réel* θ $(a' \leqq \theta \leqq b')$ *satisfassent à l'inégalité* (8), *il suffit que* $y + \beta(x)$ *appartient au segment* $T(x)$ *et qu'en même temps* θ *appartient à l'intervalle*

$$Q(x, y) \ldots . \; -\varphi(x)\,\Phi'_y(x, y+\beta(x)) < \theta - \Phi(x, y+\beta(x)) < 0. \qquad (17)$$

Démonstration. Il suit de (17)

$$\theta = \Phi(x, y+\beta(x)) - \vartheta\,\varphi(x)\,\Phi'_y(x, y+\beta(x)) \;\; (0 < \vartheta < 1),$$

donc

$$F(x, \theta) = F(x, \Phi(x, y+\beta(x))) - \vartheta\,\varphi(x)\,\Phi'_y(x, y+\beta(x)).$$
$$.\,F'_\theta(x, \Phi(x, y+\beta(x)) - \vartheta'\,\varphi(x)\,\Phi'_y(x, y+\beta(x)))$$

$(0 < \vartheta' < \vartheta)$; c'est à dire

$$0 > F(x, \theta) - y - \beta(x) > -\varphi(x) \frac{F'_\theta(x, \Phi(x, y+\beta(x)))}{F'_\theta(x, \Phi(x, y+\beta(x)))} =$$
$$= -\varphi(x) = -\beta(x) + \alpha(x),$$

d'où (8).

Lemme 3. *Si, les conditions* I *du théorème* 4A *étant remplies, pour* $x \geqq x_2$ *le nombre* y *parcourt les nombres entiers du segment*

$$R(x) \ldots . \; F(x, A) - \alpha(x) \leqq y \leqq F(x, B) - \beta(x), \quad \ldots \qquad (18)$$

où x_2, A et B désignent les nombres du III-ième remarque du § 7, alors toutes les intervalles $Q(x, y)$ (x étant supposé fixe) définies par (17) sont situées dans l'intervalle $A \leqq \theta \leqq B$.

Démonstration. Des relations (17) et (18) nous déduisons

$$\theta < \Phi(x, y + \beta(x)) \leqq \Phi(x, F(x, B)) = B$$

et en outre

$$\left.\begin{aligned} \theta > \Phi(x, y + \beta(x)) - \varphi(x)\, \Phi'_y(x, y + \beta(x)) \geqq \\ \Phi(x, F(x, A) + \varphi(x)) - \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)), \end{aligned}\right\} \quad (19)$$

comme $\Phi(x, z) - \varphi(x)\Phi'_y(x, z)$ est monotone, non, décroissante par rapport à z. Remarquons que

$$F(x, A + \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x))) =$$

$$F(x, A) + \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)) \,.$$

$$. F'_\theta(x, A + \vartheta\, \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)))\ (0 < \vartheta < 1)$$

$$\leqq F(x, A) + \varphi(x) \frac{F'_\theta(x, A + \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)))}{F'_\theta(x, \Phi(x, F(x, A) + \varphi(x)))}$$

$$= F(x, A) + \varphi(x) \frac{F'_\theta(x, A + \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)))}{F'_\theta(x, A + \varphi(x)\Phi'_y(x, F(x, A) + \vartheta'\, \varphi(x)))}$$

(d'après le théorème de la moyenne; $0 < \vartheta' < 1$)

$$\leqq F(x, A) + \varphi(x)$$

en vertu de la monotonie de F'_θ et de Φ'_y par rapport à θ, resp. à y.

Par conséquent il suit de (19)

$$\theta > \Phi(x, F(x, A + \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)))$$
$$- \varphi(x)\, \Phi'_y(x, F(x, A) + \varphi(x)) = A.$$

Lemme 4. *Si, les conditions du lemme 2 étant remplies, nous désignons par $E(x)$ l'ensemble des intervalles $Q(x, y)$ définies par (17), où y parcourt les nombres entiers du segment $R(x)$, défini dans le lemme 3 par (18), ($x \geqq x_2$), alors deux intervalles différentes de $E(x)$ n'ont pas de points communs.*

Démonstration. Comme les points terminaux à droite $\Phi(x, y + \beta(x))$ des intervalles $Q(x, y)$ constituent une suite croissante par rapport à y, il suffit de démontrer

$$\Phi(x, y + \beta(x) + 1) - \Phi(x, y + \beta(x)) > \varphi(x)\, \Phi'_y(x, y + \beta(x) + 1).$$

Or, le membre gauche est égal à

$$\Phi'_y(x, y + \beta(x) + \vartheta) > \varphi(x)\, \Phi'_y(x, y + \beta(x) + 1)\ (0 < \vartheta < 1).$$

Lemme 5. *Soit ε un nombre positif quelconque. Alors il y a un nombre $x_3 = x_3(\varepsilon, A, B) \geqq x_2$, tel que*

$$m\,E(x) > \frac{\varphi(x)}{1+\varepsilon}(B-A), \quad si\ x \geqq x_3,$$

où $mE(x)$ désigne la mesure de l'ensemble $E(x)$ du lemme 4.

Démonstration. Il résulte du lemme 4

$$m\,E(x) \geqq \sum_{\substack{y \text{ dans } R(x) \\ y \text{ entier}}} \varphi(x)\,\Phi'_y(x, y+\beta(x))$$

$$\geqq \varphi(x) \int\limits_{F(x,A)-a(x)+2}^{F(x,B)-\beta(x)} \Phi'_y(x, y+\beta(x))\,dy =$$

$$= \varphi(x)\,\{B - \Phi(x, F(x,A)+\varphi(x)+2)\}$$

$$\geqq \varphi(x)\,\{B - A - 3\,\Phi'_y(x, F(x,A)+3\vartheta)\}\ (0<\vartheta<1)$$

et alors

$$\geqq \frac{\varphi(x)}{1+\varepsilon}(B-A),\ si\ x \geqq x_3(\varepsilon, A, B).$$

Lemme 6. *Soit i, k un couple d'entiers $(0 \leqq i \leqq k-1)$, alors afin que, les définitions du lemme 4 étant employées, deux intervalles $Q(x+i, v)$ et $Q(x+k, y)$, resp. de l'ensemble $E(x+i)$ et de l'ensemble $E(x+k)$ $(x \geqq x_2)$, aient un point commun, il faut que*

$$v + \beta(x+i) - F(x+i, \Phi(x+k, y+\beta(x+k))) < \varphi(x+i). \tag{20}$$

Démonstration. D'après le lemme 3 les deux intervalles sont situées dans $A \leqq \theta \leqq B$. Considérons la distance λ de leurs points terminaux à droite, c'est à dire

$$\lambda = \Phi(x+k, y+\beta(x+k)) - \Phi(x+i, v+\beta(x+i)). \tag{21}$$

Alors

$$\Phi(x+i, v+\beta(x+i)) = \Phi(x+k, y+\beta(x+k)) - \lambda$$

et donc

$$v + \beta(x+i) = F(x+i, \Phi(x+k, y+\beta(x+k)-\lambda)$$

$$= F(x+i, \Phi(x+k, y+\beta(x+k)) +$$

$$- \lambda\,F'_0(x+i, \Phi(x+k, y+\beta(x+k)) - \vartheta\lambda)$$

où $0 < \vartheta < 1$. Maintenant distinguons deux cas.

A. Soit $\lambda \geqq 0$. Alors on a

$$\left.\begin{aligned} &|v + \beta(x+i) - F(x+i, \Phi(x+k, y+\beta(x+k)))| \geqq \\ &\qquad \geqq |\lambda|\,F'_0(x+i, \Phi(x+k, y+\beta(x+k))). \end{aligned}\right\} \tag{22 A}$$

B. Soit $\lambda \leqq 0$. Alors on a

$$\left.\begin{aligned} &|v+\beta(x+i)-F(x+i,\Phi(x+k,y+\beta(x+k)))| \leqq \\ &\qquad |\lambda| F'_\theta(x+i,\Phi(x+k,y+\beta(x+k))+|\lambda|)= \\ &\qquad\qquad =|\lambda| F'_\theta(x+i,\Phi(x+i,v+\beta(x+i))). \end{aligned}\right\} \quad (22\,\mathrm{B})$$

2. Remarquons que pour l'éxistence d'un point commun aux intervalles $Q(x+i,v)$ et $Q(x+k,y)$, il faut en vertu de (17) et de (21) que

$$|\lambda| < \varphi(x+k)\,\Phi'_y(x+k,y+\beta(x+k)),\ si\ \lambda \geqq 0, \quad . \quad (23\,\mathrm{A})$$

$$|\lambda| < \varphi(x+i)\,\Phi'_y(x+i,v+\beta(x+i)),\ si\ \lambda \leqq 0. \quad . \quad . \quad (23\,\mathrm{B})$$

Distinguons deux cas encore.

I. Soit $\lambda \geqq 0$. Alors nous tirons de (22A) et (23A)

$$|v+\beta(x+i)-F(x+i,\Phi(x+k,y+\beta(x+k)))|$$

$$<\varphi(x+k)\frac{F'_\theta(x+i,\Phi(x+k,y+\beta(x+k)))}{F'_\theta(x+k,\Phi(x+k,y+\beta(x+k)))} \leqq \varphi(x+k) \leqq \varphi(x+i).$$

II. Soit $\lambda \leqq 0$. Alors nous tirons de (22B) et (23B)

$$|v+\beta(x+i)-F(x+i,\Phi(x+k,y+\beta(x+k)))|$$

$$<\varphi(x+i)\frac{F'_\theta(x+i,\Phi(x+i,v+\beta(x+i)))}{F'_\theta(x+i,\Phi(x+i,v+\beta(x+i)))}=\varphi(x+i).$$

C. f. d.

Lemme 7. *Désignons, les définitions du lemme 4 étant employées, par $G(x+i,x+k)$ $(x \geqq x_2, 0 \leqq i \leqq k-1)$ l'ensemble constitué de toutes ces intervalles $Q(x+k,y)$ de $E(x+k)$ qui ont au moins un point commun avec au moins une des intervalles $Q(x+i,v)$ de $E(x+i)$. Alors à tout nombre positif ε' correspond un indice $x_4 = x_4(\varepsilon', A, B) \geqq x_2$, tel que pour $x \geqq x_4$ on ait*

$$\left.\begin{aligned} m\,G(x+i,x+k) \leqq \varphi(x+k)\Big\{&2(1+\varepsilon')\varphi(x+i)(B-A)+ \\ &+\int_A^B \frac{F'_\theta(x+i,t)}{F'_\theta(x+k,t)}\,dt+\frac{1}{F'_\theta(x+k,A)}\Big\} \end{aligned}\right\} \quad (24)$$

Démonstration. La fonction $F(x+i, \Phi(x+k, y+\beta(x+k)))$ est une fonction croissante par rapport à y; elle parcourt le segment $T(x+i)$, si $y+\beta(x+k)$ parcourt $T(x+k)$. Sur l'intervalle

$$F(x+k,\Phi(x+i,v+\beta(x+i)-1))-\beta(x+k) \leqq$$
$$\leqq y < F(x+k,\Phi(x+i,v+\beta(x+i)))-\beta(x+k)$$

elle croît de $v+\beta(x+i)-1$ à $v+\beta(x+i)$ et sur l'intervalle

$$F(x+k,\Phi(x+i,v+\beta(x+i)))-\beta(x+k) \leqq$$
$$y < F(x+k,\Phi(x+i,v+\beta(x+i)+1))-\beta(x+k)$$

elle croît de $v+\beta(x+i)$ à $v+\beta(x+i)+1$ (si du moins ces trois valeurs sont situées dans le domaine $T(x+i)$). Ça veut dire que si v désigne

un nombre naturel donné, tous les entiers y qui satisfont à (20) seront dans l'intervalle

$$\left.\begin{aligned} S(v,i) \ldots . F(x+k, \Phi(x+i, v+\beta(x+i)-\varphi(x+i)))-\beta(x+k) \\ <y<F(x+k, \Phi(x+i, v+\beta(x+i)+\varphi(x+i)))-\beta(x+k). \end{aligned}\right\} \quad (25)$$

Supposons maintenant que v parcourt les nombres entiers de $R(x+i)$, défini par (18). Pour que $Q(x+i, v)$ et $Q(x+k, y)$ aient des points communs, il faut d'après le lemme 6 que (20) soit rempli et donc d'après ce qui précède que l'entier y, pour le nombre v correspondant, soit situé dans $S(v, i)$. La définition de $Q(x, y)$ par (17) entraine donc immédiatement

$$m\, G(x+i, x+k) \leqq \sum_{\substack{v \text{ dans } R(x+i) \\ v \text{ entier}}} \; \sum_{\substack{y \text{ dans } S(v,i) \\ y \text{ entier}}} \varphi(x+k)\, \Phi'_y(x+k, y+\beta(x+k)). \quad (26)$$

D'après (25) la somme intérieure

$$\begin{aligned} \sum_{\substack{y \text{ dans } S(v,i) \\ y \text{ entier}}} &\leqq \varphi(x+k) \int_{F(x+k, \Phi(x+i, v+\alpha(x+i)))-\beta(x+k)}^{F(x+k, \Phi(x+i, v+\beta(x+i)+\varphi(x+i)))-\beta(x+k)} \Phi'_y(x+k, y+\beta(x+k))\, dy \\ &+\varphi(x+k)\, \Phi'_y(x+k, F(x+k, \Phi(x+i, v+\alpha(x+i)))) \\ &=\varphi(x+k)\{\Phi(x+k, F(x+k, \Phi(x+i, v+\beta(x+i)+\varphi(x+i)))) \\ &-\Phi(x+k, F(x+k, \Phi(x+i, v+\alpha(x+i))))\} \\ &+\varphi(x+k)\{F'_\vartheta(x+k, \Phi(x+i, v+\alpha(x+i)))\}^{-1} \\ &=\varphi(x+k)\{\Phi(x+i, v+\alpha(x+i)+2\varphi(x+i))-\Phi(x+i, v+\alpha(x+i))\} \\ &+\varphi(x+k)\{F'_\vartheta(x+k, \Phi(x+i, v+\alpha(x+i)))\}^{-1} \\ &\leqq 2\varphi(x+k)\, \varphi(x+i)\, \Phi'_y(x+i, v+\alpha(x+i)) \\ &+\varphi(x+k)\{F'_\vartheta(x+k, \Phi(x+i, v+\alpha(x+i)))\}^{-1}, \end{aligned}$$

c'est à dire que (26) nous apprend

$$\begin{aligned} m\, G(x+i, x+k) &\leqq 2\varphi(x+k)\, \varphi(x+i) \sum_{\substack{v \text{ dans } R(x+i) \\ v \text{ entier}}} \Phi'_y(x+i, v+\alpha(x+i)) \\ &+\varphi(x+k) \sum_{\substack{v \text{ dans } R(x+i) \\ v \text{ entier}}} \{F'_\vartheta(x+k, \Phi(x+i, v+\alpha(x+i)))\}^{-1} \end{aligned}$$

$$\begin{aligned} &\leqq 2\varphi(x+k)\, \varphi(x+i) \left\{ \int_{F(x+i, A)-\alpha(x+i)}^{F(x+i, B)-\alpha(x+i)} \Phi'_y(x+i, v+\alpha(x+i))\, dv + \Phi'_y(x+i, F(x+i, A)) \right\} \\ &+\varphi(x+k) \left\{ \int_{F(x+i, A)-\alpha(x+i)}^{F(x+i, B)-\alpha(x+i)} \{F'_\vartheta(x+k, \Phi(x+i, v+\alpha(x+i)))\}^{-1}\, dv + \{F'_\vartheta(x+k, A)\}^{-1} \right\}. \end{aligned}$$

On voit immédiatement que la première intégrale a la valeur $B-A$: en posant dans la dernière intégrale

$t=\Phi(x+i, v+\alpha(x+i))$, c' est à dire $v=F(x+i, t)-\alpha(x+i)$, nous la donnons la forme

$$\int_A^B \frac{F_\theta'(x+i, t)}{F_\theta'(x+k, t)}\, dt,$$

de sorte que

$$m\, G(x+i, x+k) \leqq 2\varphi(x+k)\varphi(x+i)\{(B-A)+\Phi_y'(x+i, F(x+i, A))\}$$
$$+\varphi(x+k)\int_A^B \frac{F_\theta'(x+i, t)}{F_\theta'(x+k, t)}\, dt+\varphi(x+k)\{F_\theta'(x+k, A)\}^{-1},$$

d'où (24), si $x \geqq x_4(\varepsilon', A, B) \geqq x_2$.

Lemme 8. *Supposons les conditions I et II du théorème 4B″ remplies. Supposons en outre les nombres ε et ε' des lemmes 5 et 7 fixés de sorte que*

$$\frac{1}{1+\varepsilon}-2(1+\varepsilon')\gamma-\delta>0, \quad \ldots \ldots \quad (27)$$

ce qui est possible d'après (12) et soit $x_5=Max(x_3, x_4)$. Désignons pour $x \geqq x_2$, $k \geqq 1$ par $H(x, k)$ l'ensemble constitué par ceux des intervalles $Q(x+k)$ de l'ensemble $E(x+k)$, défini dans le lemme 4, qui n'ont aucun point commun avec aucun des ensembles $E(x+i)$ $(0 \leqq i \leqq k-1)$ et posons $H(x, 0)=E(x)$.

Alors il y a un indice $x_6 \geqq x_5$, tel qu' à tout nombre entier $x \geqq x_6$ correspond un indice $K=K(x)$, tel que

$$\sum_{k=0}^{K} m\, H(x, k)>\delta_0(B-A), \quad \ldots \ldots \quad (28)$$

où δ_0 désigne un nombre positif, indépendant de x, K, A et B (A et B désignant les nombres de la III-ième remarque du § 7).

Démonstration. La définition de l'énsemble $G(x+i, x+k)$ entraine

$$m\, H(x, k) \geqq m\, E(x+k)-\sum_{i=0}^{k-1} m\, G(x+i, x+k)$$

$(k \geqq 1)$ et donc d'après les lemmes 5 et 7

$$m\, H(x, k) \geqq \varphi(x+k)(B-A)\left\{\frac{1}{1+\varepsilon}-2(1+\varepsilon')\sum_{i=0}^{k-1}\varphi(x+i)\right\}$$
$$-\varphi(x+k)\int_A^B\left\{\sum_{i=0}^{k-1}\frac{F_\theta'(x+i, t)}{F_\theta'(x+k, t)}\right\}dt-\varphi(x+k)\sum_{i=0}^{k-1}\frac{1}{F_\theta'(x+k, A)}$$

$$\geqq \varphi(x+k)(B-A)\left\{\frac{1}{1+\varepsilon}-2(1+\varepsilon)\sum_{i=0}^{k-1}\varphi(x+i)-\right.$$
$$\left.(1+\varepsilon'')\, \underset{A \leqq t \leqq B}{Max}\, \sum_{i=0}^{k-1}\frac{F_\theta'(x+i, t)}{F_\theta'(x+k, t)}\right\}$$

pour tout $\varepsilon'' > 0$, si $x \geqq x_6(\varepsilon'', A, B) \geqq x_5$. Choissisons ε'' assez petit pour que

$$\eta = \frac{1}{1+\varepsilon} - 2(1+\varepsilon')\gamma - (1+\varepsilon'')\delta > 0$$

ce qui est possible d'après (27). Soit $K = K(x)$ l'indice $\geqq 0$ du théorème 4B''. Si $K \geqq 1$, nous aurons pour $1 \leqq k \leqq K-1$ d'après (13) et (14)

$$m\,H(x,k) \geqq \varphi(x+k)(B-A)\left\{\frac{1}{1+\varepsilon} - 2(1+\varepsilon')\gamma - (1+\varepsilon'')\delta\right\}$$
$$= \eta\,\varphi(x+k)(B-A),$$

en outre nous aurons d'après le lemme 5

$$m\,H(x,0) \geqq \frac{\varphi(x)}{1+\varepsilon}(B-A) > \eta\,\varphi(x)(B-A).$$

Nous aurons donc

$$\sum_{k=0}^{K} m\,H(x,k) \geqq \eta(B-A)\sum_{k=0}^{K}\varphi(x+k) > \delta_0(B-A),$$

où $\delta_0 = \eta\gamma$ d'après (13). C.f.d.

2. *Fin de la démonstration.* D'après le lemme 2 un point θ de $A \leqq \theta \leqq B$ sera contenu dans l'ensemble $E_{A,B}$ des points θ pour lesquels l'inégalité (8) possède une infinité de solutions entières $x \geqq 1$, y, dès que θ est contenu dans une infinité des ensembles $E(x)$ du lemme 4, donc dès que pour tout nombre entier $x \geqq x_2$ le point θ est situé dans l'ensemble

$$L(x) = \sum_{k=0}^{\infty} E(x+k) \quad \ldots\ldots\ldots \quad (29)$$

Concluons:

$$L(x_2) \supset L(x_2+1) \supset \ldots; \quad \ldots\ldots \quad (30)$$

l'ensemble $E_{A,B}$ contiendra l'ensemble produit M des ensembles $L(x)$. Mais de (29) et la définition de l'ensemble $H(x,k)$ dans le lemme 8 nous déduisons

$$m\,L(x) \geqq \sum_{k=0}^{K} m\,H(x,k)$$

pour tout nombre entier $K \geqq 0$ et donc d'après le lemme 8 et (30)

$$m\,E_{A,B} \geqq m\,M \geqq \delta_0(B-A).$$

Or, comme les nombres A et B sont des nombres quelconques, tels que $a' \leqq A < B \leqq b'$, nous en concluons que la densité de $E_{a',b'}$ dans chaque point θ du segment $a' \leqq \theta \leqq b'$ est au moins $\delta_0 > 0$ et donc $= 1$ d'après un théorème de LEBESQUE bien connu dans la théorie de la mesure. C.f.d.

Mathematics. — *Proof of the impossibility of a just distribution of an infinite sequence of points over an interval.* By T. VAN AARDENNE–EHRENFEST. (Communicated by Prof. J. G. VAN DER CORPUT.)

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Introduction. About ten years ago Professor J. G. VAN DER CORPUT considered the question whether the distribution of an infinite sequence of points over an interval might be of a nature which he described by the terme *just*, and which is much more regular than that required of what is called a *uniform* distribution. An infinite sequence $\mathfrak{A}$ of points in an interval is said to be *justly distributed* over this interval if there exists a constant C such that for any pair of sub-intervals α, β of equal length and all n

$$|A_n(\alpha) - A_n(\beta)| \leqslant C.$$

Here $A_n(\alpha)$ and $A_n(\beta)$ denote the numbers of points of $\mathfrak{A}$ with indices not exceeding n which belong to α and β respectively.

After some attempts to construct such a sequence, Professor VAN DER CORPUT was led to the conjecture *that a just distribuion does not exist*. In what follows it will be proved that this conjecture is right.

Notations. Greek letters will denote intervals (which may be either closed, or open, or half-open, or consist of one point only). $|\alpha|$ denotes the length of α, $\alpha \subset \beta$ means that α is contained in β.

If $\mathfrak{A}$ is a finite system of (not necessarily mutually different) points, then $A(\lambda)$ denotes the number of points of $\mathfrak{A}$ contained in λ. If $a_1, a_2, \ldots$ is a sequence of points, then $A_n(\lambda)$ will denote the number of points of the initial fragment $a_1, a_2 \ldots a_n$ belonging to λ.

Theorem. To every natural number x there correspond a natural number $N(x)$ and a positive number $u(x)$ with the following property: if ι is an interval of length $|\iota| > 0$, then any finite sequence of $N(x)$ points $a_1, a_2 \ldots a_{N(x)}$ in ι has an initial fragment $a_1, a_2 \ldots a_n$ $(n \leq N(x))$ such that ι contains two intervals σ and τ for which

$$|\tau| = |\sigma| + |\iota|\, u(x) \text{ and } A_n(\sigma) \geqslant A_n(\tau) + x.$$

From this theorem it follows immediately that a just distribution does not exist, for the interval τ mentioned in the theorem contains an interval τ' of length $|\tau'| = |\sigma|$, and we have $A_n(\sigma) \geq A_n(\tau') + x$.

Proof of the theorem. Without loss of generality we suppose $|\iota| = 1$.

$N(1) = 1$ and $u(1) = \frac{1}{2}$ have the required property; for if the sequence consists of one point only, there is a (closed) sub-interval of ι of length 0

containing one point of the sequence, and there is an (open) sub-interval of length $\frac{1}{2}$ containing no point of the sequence.

Suppose that for some particular x the existence of $N = N(x)$ and $u = u(x)$ has been established. We have to show the existence of $N(x+1)$ and $u(x+1)$.

Let T be an even positive integer such that $u \geq T^{-1}$. We shall prove that the integer M and the positive number w, which are determined by the recursive relations

$$q_{1,0} = 1, \quad f_i = [^2\log(5\,x\,q_{i,0})] + 1 \quad (i = 1, \dots x)$$

$$q_{i,k+1} = (2\,q_{i,k} + 1)\,T \quad (i = 1, \dots x;\; k = 0, 1, \dots f_i - 1)$$

$$q_{i+1,0} = q_{i,f_i} + 2 \quad (i = 1, \dots x)$$

$$M = 2\,q_{x+1,0}(N + 6\,x), \quad d = \frac{1}{q_{x+1,0}}, \quad w = \frac{d}{(x+1)(5x+1)}$$

have the properties required of $N(x+1)$ and $u(x+1)$.

Suppose $\mathfrak{C}$ is a sequence of M points in ι such that each initial fragment $\mathfrak{C}'$ of $\mathfrak{C}$ has the property that

$$C'(\lambda) \leqslant C'(\mu) + x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when $|\mu| = |\lambda| + w$, $\lambda \subset \iota$, $\mu \subset \iota$. We have to show that this is absurd.

We begin by deducing some properties of $\mathfrak{C}$.

Put $M' = 2xq_{x+1,0} + 1$. Let $\mathfrak{A}$ be the system of the first M' points of $\mathfrak{C}$ and let $\mathfrak{B}$ be the sequence which is the remaining fragment of $\mathfrak{C}$.

Property 1. Each sub-interval of ι of length d contains at most $5x$ points of $\mathfrak{A}$ and at least 1 point of $\mathfrak{A}$ and N points of $\mathfrak{B}$.

Proof. $\mathfrak{A}$ is an initial fragment of $\mathfrak{C}$. Hence it follows from (1): if there were a sub-interval of ι of length d containing more than $5x$ points of $\mathfrak{A}$, then each sub-interval of length $d + w$, and a fortiori (since $w < d$) each sub-interval of length $d + d = 2d$, would contain at least $5x + 1 - x = 4x + 1$ points of $\mathfrak{A}$. Since $\frac{1}{2}\,q_{x+1,0}$ is an integer (which follows from the given formulae and from the fact that T is even), the interval ι, being of length $1 = \frac{1}{2}\,q_{x+1,0} \cdot 2d$, would contain at least

$$\tfrac{1}{2}\,q_{x+1,0}\,(4\,x + 1) = 2\,x\,q_{x+1,0} + \tfrac{1}{2}\,q_{x+1,0}$$

points of $\mathfrak{A}$, which is more than

$$M' = 2\,x\,q_{x+1,0} + 1, \text{ since } \tfrac{1}{2}\,q_{x+1,0} > 1.$$

If, on the other hand, there were a sub-interval of ι of length d containing no point of $\mathfrak{A}$ (or less than N points of $\mathfrak{B}$ and therefore less than $N + 5x$ points of $\mathfrak{A} + \mathfrak{B} = \mathfrak{C}$), it follows from (1), since $\mathfrak{A}$ and $\mathfrak{C}$ are both initial fragments of $\mathfrak{C}$, that each sub-interval of length $d - w$, and a fortiori (since $w < \frac{1}{2}\,d$) each sub-interval of length $d - \frac{1}{2}\,d = \frac{1}{2}\,d$, would

contain at most $0+x=x$ points of $\mathfrak{A}$ (or less than $N+5x+$ $+x=N+6x$ points of $\mathfrak{C}$). Hence the interval ι, being of length $1=2q_{x+1,0}\cdot\frac{1}{2}d$, would contain at most $2xq_{x+1,0}=M'-1$ points of $\mathfrak{A}$ (or less than $2q_{x+1,0}(N+6x)=M$ points of $\mathfrak{C}$). Either is impossible, and so property 1 is proved.

In what follows we shall denote by ω_r ($r=0, 1, \dots x$) a sub-interval of ι of length $q_{x-r+1,0}\,d$ with the property that

$$A(\lambda)\leqq A(\mu)+x-r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whenever $|\mu|=|\lambda|+(r+1)w$, $\lambda\subset\omega_r$, $\mu\subset\omega_r$.

By ϱ_r^k ($r=0, 1, \dots x-1$; $k=0, 1, \dots f_{x-r}$) we shall denote an interval of length $q_{x-r,k}\,d$ which is contained in an ω_r, and which has a distance not less than d from the end-points of this ω_r.

Property 2. Each ϱ_r^k contains 2^k intervals of length $q_{x-r,0}\,d$ for which

$$A(\lambda)\leqq A(\mu)+x-r-1$$

whenever $|\mu|=|\lambda|+(r+2)w$ and λ is contained in one of these 2^k sub-intervals while μ is contained in another of these intervals.

Proof. This is obvious for $k=0$. Suppose that property 2 has been verified for a particular k $(0\leq k\leq f_{x-r}-1)$. We shall verify it for $k+1$ instead of k.

Consider an interval ϱ which is a ϱ_r^{k+1}. ϱ is contained in an ω_r, to the end-points of which it has a distance not less than d. This ω_r we call ω.

Since $|\varrho|=q_{x-r,k+1}\,d>d$, the interval ϱ contains, by property 1, at least N points of $\mathfrak{B}$. Since it was supposed that $N=N(x)$ and $u=u(x)$ satisfy the requirements of the theorem, the sequence $\mathfrak{B}$ has an initial fragment $\mathfrak{B}'$ such that ϱ contains two intervals σ and τ for which

$$B'(\sigma)\geqq B'(\tau)+x \text{ and } |\tau|=|\sigma|+q_{x-r,k+1}\,d\,u.$$

Without loss of generality we assume that σ and τ do not overlap, and that σ lies to the left of τ. Since $u\geq T^{-1}$, there will exist a sub-interval τ' of τ of length

$$|\tau'|=|\sigma|+T^{-1}q_{x-r,k+1}\,d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have

$$B'(\sigma)\geqq B'(\tau')+x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By taking away from τ' on both sides intervals α and β of length

$$|\alpha|=|\beta|=\tfrac{1}{2}\{T^{-1}q_{x-r,k+1}\,d-d\}=q_{x-r,k}\,d$$

we obtain an interval γ of length

$$|\gamma|=|\tau'|-(T^{-1}q_{x-r,k+1}\,d-d)=|\sigma|+d,$$

separating α and β.

First we shall prove that

$$A(\lambda_1) \leq A(\mu_1) + x - r - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (^*)$$

whenever $|\mu_1| = |\lambda_1| + (r+2)w$ *and either*

$$\lambda_1 \subset \alpha,\ \mu_1 \subset \beta$$

or

$$\lambda_1 \subset \beta,\ \mu_1 \subset \alpha.$$

Let ν be the interval which separates λ_1 and μ_1. Put $\lambda_2 = \lambda_1 + \nu$ and $\mu_2 = \mu_1 + \nu$. Then (*) means the same thing as

$$A(\lambda_2) \leq A(\mu_2) + x - r - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (^{**})$$

We have

$$|\mu_2| = |\lambda_2| + (r+2)w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since λ_2 contains ν, and ν contains γ, we have

$$|\lambda_2| \geq |\gamma| = |\sigma| + d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The interval σ is contained in ϱ; ϱ is contained in ω and has to its end-points a distance not less than d. Therefore σ is contained in ω and has to its end-points a distance not less than d. Since, by property 1, each sub-interval of ι of length d contains at least 1 point of $\mathfrak{A}$, there is a point a of $\mathfrak{A}$ belonging to ω which is situated to the left of σ, and whose distance from σ is smaller than d.

Now consider the *open* interval $\varkappa$ which has its left-hand end-point in a and for which

$$|\varkappa| = |\lambda_2| + (r+1)w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By (5) we have

$$|\varkappa| = |\mu_2| - w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $\bar{\varkappa}$ is the *closed* interval with the same end-points as $\varkappa$, we have

$$A(\varkappa) \leq A(\bar{\varkappa}) - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

since $\varkappa$ is open, and since at least one point of $\mathfrak{A}$ coincides with an end-point of $\bar{\varkappa}$.

The interval σ is contained in $\varkappa$, for, by (6) and (7), we have $|\varkappa| > |\sigma| + d$, and the left-hand end-point a of $\varkappa$, which is situated to the left of σ, has a distance to it which is smaller than d. Therefore we have by (4), observing that μ_2 in contained in τ',

$$B'(\varkappa) \geq B'(\sigma) \geq B'(\tau') + x \geq B'(\mu_2) + x \quad . \quad . \quad . \quad . \quad (10)$$

The left-hand end-point a of $\varkappa$ belongs to ω and is situated to the left of σ and a fortiori to the left of τ', which was supposed to be situated to the right of σ; the length of $\varkappa$ is, by (8), smaller than the length of τ', since μ_2 is contained in τ'; since τ' is contained in ω, it follows that the right-hand end-point of $\varkappa$ belongs also to ω. Hence the interval $\varkappa$ is contained in ω.

Since $\varkappa$ and λ_2 are contained in ω, which is an ω_r, it follows from (2) and (7) that

$$A(\lambda_2) \leqslant A(\varkappa) + x - r \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

Since $\bar{\varkappa}$ and μ_2 are contained in ι, and since $\mathfrak{A} + \mathfrak{B}'$ is an initial fragment of $\mathfrak{C}$, it follows from (1) and (8) that

$$A(\bar{\varkappa}) + B'(\bar{\varkappa}) \leqslant A(\mu_2) + B'(\mu_2) + x. \quad . \; . \; . \; . \; . \quad (12)$$

From (9), (10), (11) and (12) follows (**), and so (*) is proved.

Each of the intervals α and β is a ϱ_r^k; for

$$|\alpha| = |\beta| = q_{x-r,k}\, d$$

and α and β are contained in ϱ, and hence they are contained in ω, which is an ω_r, and have to its end-points a distance not smaller than d. Since we supposed that property 2 had been verified for our particular value of k, it follows that α contains 2^k intervals α_j $(j = 1, \dots 2^k)$ and β contains 2^k intervals β_j $(j = 1, \dots 2^k)$ such that $|\alpha_j| = |\beta_j| = q_{x-r,0}\, d$, and that

$$A(\lambda) \leqslant A(\mu) + x - r - 1. \quad . \; . \; . \; . \; . \; . \; . \quad (13)$$

whenever $|\mu| = |\lambda| + (r+2)w$ and either

$$\lambda \subset \alpha_{j_1}, \mu \subset \alpha_{j_2}, j_1 \neq j_2$$

or

$$\lambda \subset \beta_{j_1}, \mu \subset \beta_{j_2}, j_1 \neq j_2.$$

From (*) it follows that (13) holds also whenever $|\mu| = |\lambda| + (r+2)w$ and either

$$\lambda \subset \alpha_{j_1}, \mu \subset \beta_{j_2}$$

or

$$\lambda \subset \beta_{j_1}, \mu \subset \alpha_{j_2}.$$

Hence the 2^{k+1} sub-intervals $\alpha_1, \alpha_2 \dots \alpha_{2^k}$, $\beta_1, \beta_2 \dots \beta_{2^k}$ of ϱ satisfy the requirements. Thus property 2 is verified for $k+1$ instead of k.

Property 3. For each of the values $r = 0, 1, \dots x$ there exists an ω_r.

Proof. This is obvious for $r = 0$, since ι itself is an ω_0. Therefore we suppose that there exists an ω_r for a particular value of r $(0 \leq r \leq x-1)$. This ω_r we call ω. We have to show that there exists an ω_{r+1}.

The interval ϱ which we obtain by taking away from ω on both sides intervals of length d, has the length $q_{x-r+1,0}\, d - 2d = q_{x-r, f_{x-r}}\, d$.

Hence ϱ is a $\varrho_r^{f_{x-r}}$. From property 2 with $k = f_{x-r}$ it follows that ϱ contains $2^{f_{x-r}}$ sub-intervals α_j $(j = 1, \dots 2^{f_{x-r}})$ of length $|\alpha_j| = q_{x-r,0}\, d$ such that

$$A(\lambda) \leqslant A(\mu) + x - r - 1 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

whenever

$$|\mu| = |\lambda| + (r+2)w, \lambda \subset \alpha_{j_1}, \mu \subset \alpha_{j_2}, j_1 \neq j_2.$$

We shall prove that among these $2^{f_{x-r}}$ intervals α_j, being of length $q_{x-r,0}\, d$, there is at least one interval which is an ω_{r+1}.

If this were not true, each interval α_j would contain two intervals σ_j and τ_j for which $|\tau_j| = |\sigma_j| + (r+2)w$ and $A(\sigma_j) > A(\tau_j) + x - r - 1$. We have $A(\sigma_j) > A(\tau_j)$, since $x - r - 1 \geq 0$. By property 1 each sub-interval of ι of length d contains at most $5x$ points of $\mathfrak{A}$, and therefore each α_j, being of length $q_{x-r,0}\, d$, contains at most $5x \cdot q_{x-r,0}$ points of $\mathfrak{A}$. Therefore $5x\, q_{x-r,0} \geq A(\sigma_j) > A(\tau_j) \geq 0$, and so $A(\sigma_j)$ can take *only* the $5x\, q_{x-r,0}$ values $1, 2, \ldots 5x\, q_{x-r,0}$. On the other hand, it follows from $f_{x-r} = [{}^2\log (5x\, q_{x-r,0})] + 1$ that $2^{f_{x-r}} > 5x\, q_{x-r,0}$ and hence the number of intervals α_j is *greater* than $5x\, q_{x-r,0}$.

Therefore there are two different intervals α_{j_1} and α_{j_2} ($j_1 \neq j_2$) such that $A(\sigma_{j_1}) = A(\sigma_{j_2})$. Without loss of generality we suppose that $|\sigma_{j_2}| \geq |\sigma_{j_1}|$ and hence $|\tau_{j_2}| \geq |\sigma_{j_1}| + (r+2)w$ while $A(\sigma_{j_1}) = A(\sigma_{j_2}) > A(\tau_{j_2}) + x - r - 1$. The interval τ_{j_2} contains an interval τ'_{j_2} of length

$$|\tau'_{j_2}| = |\sigma_{j_1}| + (r+2)\, w$$

and we have

$$A(\sigma_{j_1}) > A(\tau'_{j_2}) + x - r - 1,\ \sigma_{j_1} \subset \alpha_{j_1},\ \tau'_{j_2} \subset \alpha_{j_2},\ j_1 \neq j_2.$$

This contradicts (14).

It follows that among the intervals α_j there is at least one which is an ω_{r+1}, and thus property 3 is proved.

From property 3 with $r = x$ it follows that there exists a sub-interval ω of ι which is an ω_x. From the definition of ω_x it follows that ω has the length $q_{1,0}\, d = d$ and further that

$$A(\lambda) \leqq A(\mu) \qquad (15)$$

whenever $|\mu| = |\lambda| + (x+1)w$, $\lambda \subset \omega$, $\mu \subset \omega$.

This is absurd since ω, being of length d, contains by property 1 at least 1 and at most $5x$ points of $\mathfrak{A}$; it will therefore contain a (closed) interval λ of length 0 to which belongs at least 1 point of $\mathfrak{A}$, and an (open) interval μ of length $\dfrac{d}{5x+1} = (x+1)w$ containing no point of $\mathfrak{A}$; hence

$$A(\lambda) > A(\mu),\ |\mu| = |\lambda| + (x+1)w,$$

which contradicts (15).

Therefore the sequence $\mathfrak{C}$ does not exist, and so the theorem is proved.

Mathematics. — *An elementary inequality.* By C. VISSER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 27, 1945.)

§ 1. Suppose that we have a certain number of rows of n elements

$$a_1, a_2, \ldots, a_n \qquad (n > 1).$$

The elements may be arbitrary objects. Suppose that the number of mutually different rows is s. Two rows are called different when they differ in at least one element occurring at the same place.

Let us omit from each of the given rows the i-th element, thus obtaining a number of rows of $n-1$ elements. Let s_i denote the number of different rows among these. We shall prove that

$$s_1\, s_2 \ldots . s_n \geqq s^{n-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with equality if and only if the given rows can be obtained from n sets of elements by taking all rows $a_1, a_2, \ldots, a_n$, where a_1 belongs to the first set, a_2 to the second set, $\ldots, a_n$ to the n-th set.

§ 2. We shall prove the inequality (1) by induction. It is clear that it is true for $n=2$. Suppose that $n > 2$, and that the inequality is true when n is replaced by $n-1$.

Let

$$a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}$$

denote all mutually different elements a_n occurring at the last place in the given rows. Let $E^{(p)}$ denote the set of all rows which have $a_n^{(p)}$ as their last element. We shall indicate the numbers s, s_i for the set $E^{(p)}$ by $\sigma^{(p)}, \sigma_i^{(p)}$.

We have

$$s_i = \sigma_i^{(1)} + \sigma_i^{(2)} + \ldots . + \sigma_i^{(k)} \text{ for } i = 1, 2, \ldots, n-1\,; \quad . \quad . \quad (2)$$

$$s_n \geqq \sigma_n^{(p)} = \sigma^{(p)} \text{ for } p = 1, 2, \ldots, k\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$s = \sigma^{(1)} + \sigma^{(2)} + \ldots . + \sigma^{(k)}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and by the inductive hypothesis

$$\sigma_1^{(p)}\, \sigma_2^{(p)} \ldots . \sigma_{n-1}^{(p)} \geqq {\sigma^{(p)}}^{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, by HÖLDER's inequality,

$$\prod_{i=1}^{n-1} (\sigma_i^{(1)} + \sigma_i^{(2)} + \ldots . + \sigma_i^{(k)})^{\frac{1}{n-1}} \geqq \sum_{p=1}^{k} (\sigma_1^{(p)}\, \sigma_2^{(p)} \ldots . \sigma_{n-1}^{(p)})^{\frac{1}{n-1}}.$$

Hence, by (2) and (5),

$$(s_1\, s_2 \ldots s_{n-1})^{\frac{1}{n-1}} \geqq \sigma^{(1)^{\frac{n-2}{n-1}}} + \sigma^{(2)^{\frac{n-2}{n-1}}} + \ldots + \sigma^{(k)^{\frac{n-2}{n-1}}}$$

and so, by (3) and (4),

$$\begin{aligned}(s_1\, s_2 \ldots s_{n-1}\, s_n)^{\frac{1}{n-1}} &\geqq \sigma^{(1)^{\frac{n-2}{n-1}}}\sigma^{(1)^{\frac{1}{n-1}}} + \sigma^{(2)^{\frac{n-2}{n-1}}}\sigma^{(2)^{\frac{1}{n-1}}} + \ldots + \sigma^{(k)^{\frac{n-2}{n-1}}}\sigma^{(k)^{\frac{1}{n-1}}}\\ &= \sigma^{(1)} + \sigma^{(2)} + \ldots + \sigma^{(k)}\\ &= s.\end{aligned}$$

This proves (1). If there is equality, then we must have *first*

$$\sigma_n^{(p)} = s_n$$

for all $p = 1, 2, \ldots, k$. This implies that the rows obtained from the set $E^{(p)}$ by omitting the n-th element $a_n^{(p)}$ from its rows are the same set for each p. *Secondly* there must be equality (with $n-1$ instead of n) for this set. It follows, sind the statement concerning the sign of equality is correct for $n = 2$, that the given set of rows is of the nature described at the end of the preceding section. On the other hand, it is clear that for any such set of rows there is equality.

§ 3. There is a more general inequality of which (1) is a particular case. Let $i_1, i_2, \ldots, i_r$ be r of the indices $1, 2, \ldots, n\, (1 \leqq r < n)$. Let us omit from each of the given rows the elements having these indices, thus obtaining a number of rows of $n-r$ elements. Denote the number of different rows among these by $s_{i_1\, i_2 \ldots i_r}$. Then

$$\Pi\, s_{i_1\, i_2 \ldots i_r} \geqq s^{\frac{(n-1)(n-2)\ldots(n-r)}{1.2.\,\ldots\, r}}, \quad \ldots \ldots \quad (6)$$

where the product must be extended over all possible combinations $i_1\, i_2 \ldots i_r$. The case of equality is the same as for (1). It is not difficult to see that (6) can be proved by successive application of (1).

§ 4. We next state a geometric application of (1). Let $x_1, x_2, \ldots, x_n$ be rectangular coordinates in an n-dimensional space. Suppose we have a set of s different points in the space. We project these points on the $(n-1)$-dimensional coordinate spaces $x_i = 0$. Let there be s_i projections on the space $x_i = 0$. It is understood that each point which is a projection is counted only once. Then we have

$$s_1\, s_2 \ldots s_n \geqq s^{n-1}.$$

There is equality if and only if the given set consists of all points $(a_1, a_2, \ldots, a_n)$ obtained from n finite sets of real numbers by taking a_1 in the first set, a_2 in the second set, $\ldots, a_n$ in the n-th set.

There is, of course, an analogous geometric application of (6).

§ 5. Finally we discuss an extension of the inequality (1) which concerns the measures of a set of points E in an n-dimensional space with rectangular coordinates $x_1, x_1, \ldots, x_n$ and its projections E_i on the $(n-1)$-dimensional coordinate spaces $x_i = 0$. In order to avoid the possible occurrence of non-measurable sets, we shall assume that E is either a closed or an open set. Further we suppose that E is bounded. It will be clear, however, from what follows that it is possible to make much milder restrictions. Let e denote the measure of E, and e_i the $(n-1)$-dimensional measure of E_i. Then

$$e_1 e_2 \ldots e_n \geqq e^{n-1}. \qquad (7)$$

It is a simple matter to derive this from the result of the preceding section, but it will be difficult to find in this way when there is equality. Therefore we shall give a direct proof.

It is clear that (7) is true for $n = 2$. Suppose that $n > 2$, and that (7) is true when n is replaced by $n-1$.

Consider the set of all points of E for which $x_n = \lambda$. Let $\varepsilon(\lambda)$ denote its $(n-1)$-dimensional measure and let $\varepsilon_i(\lambda)$ denote the $(n-2)$-dimensional measure of its projection on the space $x_i = 0$ $(i = 1, 2, \ldots, n-1)$. We have

$$e_i = \int \varepsilon_i(\lambda)\, d\lambda \ \text{ for } i = 1, 2, \ldots, n-1; \qquad (8)$$

$$e_n \geqq \varepsilon(\lambda) \ \text{ for all } \lambda; \qquad (9)$$

$$e = \int \varepsilon(\lambda)\, d\lambda; \qquad (10)$$

and by the inductive hypothesis

$$\varepsilon_1(\lambda)\, \varepsilon_2(\lambda) \ldots \varepsilon_{n-1}(\lambda) \geqq \varepsilon(\lambda)^{n-2} \qquad (11)$$

Now, by HÖLDER's inequality for integrals,

$$\prod_{i=1}^{n-1} \left(\int \varepsilon_i(\lambda)\, d\lambda \right)^{\frac{1}{n-1}} \geqq \int [\varepsilon_1(\lambda)\, \varepsilon_2(\lambda) \ldots \varepsilon_{n-1}(\lambda)]^{\frac{1}{n-1}}\, d\lambda.$$

Hence, by (8) and (11),

$$(e_1 e_2 \ldots e_{n-1})^{\frac{1}{n-1}} \geqq \int \varepsilon(\lambda)^{\frac{n-2}{n-1}}\, .\, d\lambda.$$

From this it follows, by (9) and (10), that

$$\begin{aligned}(e_1 e_2 \ldots e_{n-1}\, e_n)^{\frac{1}{n-1}} &\geqq \int \varepsilon(\lambda)^{\frac{n-2}{n-1}}\, \varepsilon(\lambda)^{\frac{1}{n-1}}\, d\lambda \\ &= \int \varepsilon(\lambda)\, d\lambda \\ &= e.\end{aligned}$$

This proves (7). A reasoning analogous to that at the end of section 2 shows that there is equality if and only if E is, a set of n-dimensional measure zero being neglected, the product of n one-dimensional sets on the n coordinate axes.

It is hardly necessary to mention that there is an analogous continuous extension of (6) concerning the $(n-r)$-dimensional measures of the projections of E on the coordinate spaces $x_{i_1} = x_{i_2} = \ldots = x_{i_r} = 0$.

Remark. For $n = 3$ it was proved by MINKOWSKI that for a *convex* set E

$$e_1 + e_2 + e_3 \geqq 3\, e^{\frac{2}{3}},$$

with equality if and only if E is a cube with its edges parallel to the coordinate axes[1]). The corresponding inequality for arbitrary n is

$$e_1 + e_2 + \ldots + e_n \geqq n\, e^{\frac{n-1}{n}} \quad \ldots \ldots \quad (12)$$

It is true for an arbitrary set E. Indeed, by the inequality of the arithmetic and geometric means, (12) is a consequence of (7). There is equality if and only if E is, a set of measure zero being neglected, the product of n sets of equal one-dimensional measure on the coordinate axes.

It is possible, also, to derive (7) from (12). Consider the set E^1 consisting of all points $(e_1 x_1, e_2 x_2, \ldots, e_n x_n)$, where $(x_1, x_2, \ldots, x_n)$ is a point in E. The measure of E^1 is $e . e_1 e_2 \ldots e_n$, and the $(n-1)$-dimensional measures of its projections on the $(n-1)$-dimensional coordinate spaces are all equal to $e_1 e_2 \ldots e_n$, and so, by (12),

$$n . e_1 e_2 \ldots e_n \geqq n\, e^{\frac{n-1}{n}},$$

and therefore

$$e_1 e_2 \ldots e_n \geqq e^{n-1},$$

with equality when E^1 satisfies the condition for equality in (12), which, as is seen at once, means that E is, but for a set of measure zero, the product of n sets on the coordinate axes.

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[1]) H. MINKOWSKI, Volumen und Oberfläche, Mathematische Annalen **57** (1903), pp. 447—495.

Mathematics. — *A simple proof of certain inequalities concerning polynomials.* By C. VISSER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 27, 1945.)

1. *Introduction.* The following theorem is due to TCHEBYCHEF.

Theorem 1. If $P(x)$ is a polynomial of degree n, and the coefficient of x^n is 1, then

$$\underset{-1 \leq x \leq 1}{Max} |P(x)| \geq \frac{1}{2^{n-1}}.$$

There is equality if and only if

$$P(x) = \frac{1}{2^{n-1}} \cos nt, \; x = \cos t.$$

This theorem is well-known. It is extensively dealt with in PÓLYA und SZEGÖ, Aufgaben und Lehrsätze aus der Analysis, 2nd volume, where further literature is indicated.

Another inequality, to which I was led, some years ago, while working at the interesting problem of Mrs. T. VAN AARDENNE–EHRENFEST in the Wiskundige Opgaven 18, Problem No. 1, is

Theorem 2. If $P(x)$ is a polynomial of degree n, and the coefficient of x^n is 1, then

$$\int_{-1}^{1} |P(x)|\, dx \geq \frac{1}{2^{n-1}}.$$

There is equality if and only if

$$P(x) = \frac{1}{2^n} \frac{\sin (n+1) t}{\sin t}, \; x = \cos t.$$

This inequality seems to be rather unknown. I am indebted to Professor KOKSMA of Amsterdam for calling my attention to the fact that the determination of the minimum of the integral of the absolute value over the interval $(-1, 1)$ for polynomials of degree n and with highest coefficient 1 was put forward as a problem by KORKIN and ZOLOTAREF in the Nouvelles

Annales Mathématiques of 1873, and that a solution was given in recent years by V. BRZECKA, Sur un problème d'extrémum (in Russian), Comm. Inst. Sci. Math. Mécan. Univ. Kharkoff etc., IV S., 16, pp. 33—44. Later I found that a proof of the inequality is implicitly contained in a paper by STIELTJES in 1876, De la représentation approximative d'une fonction par une autre, Oeuvres complètes, 1st volume, pp. 11—20.

In what follows I shall give a simple proof of both Theorem 1 and 2, and some generalizations.

2. *Proof of Theorem 1.* Since

$$P(\cos t) = \cos^n t + \ldots. = \frac{1}{2^{n-1}}(\cos n t + \ldots.),$$

Theorem 1 will follow from

Theorem 1a. If

$$C(t) = \cos n t + a_1 \cos (n-1) t + \ldots. + a_n$$

with arbitrary complex coefficients $a_1, \ldots, a_n$, then

$$Max\,|\,C(t)\,| \geqq 1.$$

There is equality if and only if $C(t) = \cos nt$.

Proof. Put $\alpha = \frac{\pi}{n}$. Then for any t

$$\sum_{l=0}^{2n-1} (-1)^l e^{i k (t+l\alpha)} = \begin{cases} 0 \text{ for } k = 0, 1, \ldots, n-1 \\ 2 n e^{int} \text{ for } k = n. \end{cases} \quad \ldots \quad (1)$$

Hence

$$\sum_{l=0}^{2n-1} (-1)^l \cos k (t + l\alpha) = \begin{cases} 0 \text{ for } k = 0, 1, \ldots, n-1 \\ 2 n \cos n t \text{ for } k = n \end{cases}$$

It follows that

$$\sum_{l=0}^{2n-1} (-1)^l C(t + l\alpha) = 2 n \cos n t.$$

Putting $t = 0$, we infer from this that

$$\underset{l=0,1,\ldots,2n-1}{Max} |\,C(l\alpha)\,| \geqq 1.$$

There is equality if and only if $(-1)^l C(l\alpha)$ is equal to 1 for all l. Then $C(t) - \cos nt$ must vanish identically, being a trigonometric polynomial of order $n-1$, and vanishing $2n$ times in the interval $0 \leqq t < 2\pi$.

3. *Proof of Theorem 2.* Since

$$\int_{-1}^{1} | P(x) | \, dx = \int_{0}^{\pi} | P(\cos t) | \sin t \, dt$$

$$= \frac{1}{2^{n-1}} \int_{0}^{\pi} | \cos n\,t + \ldots . | \sin t \, dt$$

$$= \frac{1}{2^{n-1}} \int_{0}^{\pi} | \cos n\,t \sin t + \ldots | \, dt$$

$$= \frac{1}{2^{n}} \int_{0}^{\pi} | \sin (n+1)\, t + \ldots . | \, dt.$$

Theorem 2 will follow from

Theorem 2a. If

$$S(t) = \sin n\,t + b_1 \sin (n-1)\, t + \ldots . + b_n$$

with arbitrary complex coefficients $b_1, \ldots, b_n$, *then*

$$\int_{0}^{2\pi} | S(t) | \, dt \geqq 4.$$

There is equality if and only if $S(t) = \sin nt$.

Proof. It follows from (1) that

$$\sum_{l=0}^{2n-1} (-1)^l \, S(t + l\,a) = 2\,n \sin nt.$$

Integrating over $0 \leqq t \leqq a$, we find

$$\sum_{l=0}^{2n-1} \int_{l\,a}^{(l+1)a} (-1)^l \, S(t) \, dt = 4.$$

It results that

$$\int_{0}^{2\pi} | S(t) | \, dt \geqq 4$$

and that there is equality if and only if $(-1)^l S(t)$ is non-negative on all intervals $la < t < (l+1)a$. Then $S(t)$ has $2n$ zeros in the points la, so that $S(t) - \sin nt$ must vanish identically, being of order $n-1$, and having $2n$ zeros in the interval $0 \leqq t < 2\pi$.

4. *Analogous inequalities for polynomials of a complex variable.*

Theorem 3. If

$$f(z) = a_0 + a_1 z + \dots a_n z^n$$

is a polynomial of the complex variable z with arbitrary complex coefficients, then

$$\underset{|z| \leqq 1}{Max} |f(z)| \geqq |a_0| + |a_n|.$$

There is equality if and only if $f(z) = a_0 + a_n z^n$.

Proof. Put $\omega = e^{i\frac{2\pi}{n}}$ Then

$$\sum_{l=0}^{n-1} f(z\,\omega^l) = n\,a_0 + n\,a_n\,z^n. \qquad \dots\dots\dots \quad (2)$$

For some z on $|z| = 1$, say z_0, this becomes $n(|a_0| + |a_n|)e^{i\alpha}$ with real α. It follows that

$$\underset{l=0,1,\dots,n-1}{Max} |f(z_0\,\omega^l)| \geqq |a_0| + |a_n|.$$

There is equality if and only if $f(z_0\omega^l) = a_0 + a_n z_0^n$ for all $l = 0, 1, \dots, n-1$. Then, obviously, $f(z) = a_0 + a_n z^n$.

Theorem 4. If

$$f(z) = a_0 + a_1 z + \dots + a_n z^n$$

is a polynomial of the complex variable z, then

$$\int_0^{2\pi} |f(e^{it})|\,dt \geqq 4(|a_0| + |a_n|).$$

There is equality if and only if $f(z) = a_0 + a_n z^n$ with $|a_0| = |a_n|$.

Proof. Putting $z = e^{it}$, we find from (2) that

$$\sum_{l=0}^{n-1} e^{-\frac{int}{2}} f\left(e^{i\left(t+l\frac{2\pi}{n}\right)}\right) = n\,a_0\,e^{-\frac{int}{2}} + n\,a_n\,e^{\frac{int}{2}}$$

Integrating over a suitably chosen interval $t_0 \leq t \leq t_0 + \frac{2\pi}{n}$, we have

$$\sum_{l=0}^{n-1} \int_{t_0+l\frac{2\pi}{n}}^{t_0+(l+1)\frac{2\pi}{n}} (-1)^l\,e^{-\frac{int}{2}} f(e^{it})\,dt = 4(|a_0| + |a_n|)\,e^{i\beta}.$$

with real β. It follows that

$$\int_0^{2\pi} |f(e^{it})|\,dt \geqq 4(|a_0| + |a_1|).$$

It is necessary for equality that

$$(-1)^l\, e^{-\frac{int}{2}} f(e^{it})$$

has the same argument β on all intervals $t_0 + l\frac{2\pi}{n} < t < t_0 + (l+1)\frac{2\pi}{n}$

This involves the vanishing of $f(e^{it})$ in all points $t_0 + l\frac{2\pi}{n}$, and so $f(z) = a_n(z^n - e^{int_0})$, i.e. $f(z) = a_0 + a_n z^n$ with $|a_0| = |a_n|$. That, conversely, in this case there is equality is easily recognized.

5. *General trigonometric polynomials.*

Theorem 5. If

$$F(t) = \sum_{k=-n}^{n} A_k e^{ikt}$$

is a trigonometric polynomial with arbitrary coefficients, then

$$Max\,|F(t)| \geqq |A_{-n}| + A_n|,$$

with equality if and only if

$$F(t) = A_{-n} e^{-int} + A_n e^{int},$$

and

$$\int_0^{2\pi} |F(t)|\,dt \geqq 4(|A_{-n}| + |A_n|),$$

with equality if and only if

$$F(t) = A_{-n} e^{-int} + A_n e^{int}$$

with $|A_{-n}| = |A_n|$.

Proof. Put

$$f(z) = z^n \sum_{k=-n}^{n} A_k z^k.$$

Then $f(z)$ is a polynomial of the complex variable z. Application of Theorems 3 and 4 yields Theorem 5.

Remark. For a trigonometric polynomial

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\,t + b_k \sin kt)$$

with real coefficients, the inequalities become

$$Max\,|\,F(t)\,| \geqq \sqrt{a_n^2 + b_n^2},$$

with equality if and only if $F(t) = a_n \cos nt + b_n \sin nt$, and

$$\int_0^{2\pi} |\,F(t)\,|\,dt \geqq 4\sqrt{a_n^2 + b_n^2}\,,$$

also with equality if and only if $F(t) = a_n \cos nt + b_n \sin nt$. The first of these is well-known; see e.g. PÓLYA und SZEGÖ, Aufgaben und Lehrsätze, 2nd volume.

Applied Mechanics. — *On the state of stress in perforated strips and plates.* (5th communication.) By K. J. SCHULZ. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of October 27, 1945.)

(This 5th and the following 6th communication have been retarded by the war. Unfortunately it is subject to severe doubt whether the author is still in life. Being a jew he has been transported first to a Dutch camp and later on to Germany or elsewhere.)

8. *The symmetrical strip with one row of holes subject to uniform flexure.* We now pass on to the consideration of the uniformly bent strip, the treatment of which will be essentially the same as that of the strip subject to tension. The only difference consists in that the state of stress in the bent strip will be anti-symmetrical with respect to the centre-line of the holes (the y-axis) whereas the state of stress of the strip under tension was symmetrical with respect to this line. This means with respect to the stress function F (3, 20), related to the centres, that the constants C_0, C_{2s}, D_{2s} $(s \geqq 1)$ must be suppressed and that this function now will be represented by

$$F = \sum_{s=0}^{\infty} (C_{2s+1}\, U_{\sigma_1 2s+1} + D_{2s+1}\, \bar{U}_{\tau_1 2s+1}). \quad . \quad . \quad . \quad . \quad (1)$$

The corresponding stresses along the circular boundaries I are given by (comp. (3, 16))

$$\left.\begin{aligned}
\sigma_r &= \sum_{n=0}^{\infty} \bar{C}_{2n+1} \sin(2n+1)\varphi + \\
&\qquad + \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, h_{2n+1}^{2s+1} - \bar{D}_{2s+1}\, i_{2n+1}^{2s+1}) \sin(2n+1)\varphi, \\
\tau_{r\varphi} &= \sum_{n=0}^{\infty} \bar{D}_{2n+1} \cos(2n+1)\varphi - \\
&\qquad - \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, j_{2n+1}^{2s+1} - \bar{D}_{2s+1}\, k_{2n+1}^{2s+1}) \cos(2n+1)\varphi, \\
\sigma_\varphi &= \sum_{n=0}^{\infty} (C_{2n+1} + 2 D_{2n+1}) \sin(2n+1)\varphi + \\
&+ \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} [(C_{2s+1}(h_{2n+1}^{2s+1} + 2 j_{2n+1}^{2s+1}) - \bar{D}_{2s+1}(i_{2n+1}^{2s+1} + 2 k_{2n+1}^{2s+1})] \sin(2n+1)\varphi
\end{aligned}\right\} \quad (2)$$

those along the straight boundaries II and III by (comp. (4, 18))

$$\left.\begin{aligned}
\sigma_z &= \pm \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, h_n'^{\,2s+1} - \bar{D}_{2s+1}\, i_n'^{\,2s+1}) \cos 2\pi n \frac{y}{b}, \\
\tau_{yz} &= + \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, j_n'^{\,2s+1} - D_{2s+1}\, k_n'^{\,2s+1}) \sin 2\pi n \frac{y}{b}, \\
\sigma_y &= \pm \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} [(C_{2s+1}(h_n'^{\,2s+1} - 2 j_n'^{\,2s+1}) - D_{2s+1}(i_n'^{\,2s+1} - 2 k_n'^{\,2s+1})] \cos 2\pi n \frac{y}{b}.
\end{aligned}\right\} \quad (3)$$

(The upper signs in (3) relate to boundary II, the lower ones to boundary III).

The anti-symmetrical character of our problem asserts itself as well with respect to the stress-functions F' related to the boundaries II and III, which now have to be provided with opposite signs

$$F' = \pm \sum_{s=1}^{\infty} (C_s' F_{\sigma s}' + D_s' F_{\tau s}'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At their "own" boundary these stress-functions give rise to the following stresses (comp. (7, 5))

$$\left.\begin{aligned}
\sigma_z &= \pm \sum_{n=1}^{\infty} C_n' \cos 2\pi n \frac{y}{b}, \\
\tau_{yz} &= + \sum_{n=1}^{\infty} D_n' \sin 2\pi n \frac{y}{b}, \\
\sigma_y &= \pm \sum_{n=1}^{\infty} (C_n' + 2 D_n') \cos 2\pi n \frac{y}{b}.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

whereas the stresses at the non-corresponding boundaries are given by (comp. (7, 6))

$$\left.\begin{aligned}
\sigma_z &= \mp \sum_{n=1}^{\infty} (C_n' p_n' + D_n' q_n') \cos 2\pi n \frac{y}{b}, \\
\tau_{yz} &= - \sum_{n=1}^{\infty} (C_n' r_n' + D_n' t_n') \sin 2\pi n \frac{y}{b}, \\
\sigma_y &= \mp \sum_{n=1}^{\infty} [C_n' (p_n' - 2 r_n') + D_n' (q_n' - 2 t_n')] \cos 2\pi n \frac{y}{b}.
\end{aligned}\right\} \quad . \quad (6)$$

(the upper signs relating to boundary II, the lower signs to boundary III).

The stresses occurring at any of the boundaries I in consequence of the *joint* functions (4) are obtained from (6, 9) by suppressing all terms with even suffixes and by doubling all terms with odd ones. They are represented by

$$\left.\begin{aligned}
\sigma_r &= + \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} (C_s' p_{2n+1}^s + D_s' q_{2n+1}^s) \sin (2n+1)\varphi, \\
\tau_{r\varphi} &= - \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} (C_s' r_{2n+1}^s + D_s' t_{2n+1}^s) \cos (2n+1)\varphi, \\
\sigma_\varphi &= + \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} [C_s' (p_{2n+1}^s - 2 r_{2n+1}^s) + D_s' (q_{2n+1}^s - 2 t_{2n+1}^s)] \sin (2n+1)\varphi.
\end{aligned}\right\} \quad (7)$$

If finally the stresses $\sigma_r^{(0)}$, $\tau_{r\varphi}^{(0)}$, $\sigma_\varphi^{(0)}$ at the boundary I, and $\sigma_z^{(0)}$, $\tau_{yz}^{(0)}$, $\sigma_y^{(0)}$ at the boundaries II and III, as occurring in the unperforated strip (and caused by the uniform flexure) are represented by the Fourier series

$$\sigma_r^{(0)} = -\sum_{n=0}^{\infty} \overline{C}_{2n+1}^{(0)} \sin(2n+1)\varphi, \quad \tau_{r\varphi}^{(0)} = -\sum_{n=0}^{\infty} \overline{D}_{2n+1}^{(0)} \cos(2n+1)\varphi \;. \quad (8)$$

and

$$\sigma_z^{(0)} = \mp \sum_{n=1}^{\infty} C_n'^{(0)} \cos 2\pi n \frac{y}{b}, \quad \tau_{yz}^{(0)} = -\sum_{n=1}^{\infty} D_n'^{(0)} \sin 2\pi n \frac{y}{b} \;. \quad . \quad (9)$$

the *total* stresses occurring at the boundaries of the perforated strip can be found, by summing up the corresponding stresses (2), (7) and (8) for boundary I) resp. the corresponding stresses (3), (5), (6) and (9) for the boundaries II and III.

The requirement that all normal and tangential stresses shall disappear at the joint boundaries leads to the following infinite system of linear equations

$$\left.\begin{aligned}
\overline{C}_{2n+1} &= \overline{C}_{2n+1}^{(0)} - \Big[\sum_{s=0}^{\infty} [\overline{C}_{2s+1} h_{2n+1}^{2s+1} - \overline{D}_{2s+1} i_{2n+1}^{2s+1}) + \\
&\qquad + \sum_{s=1}^{\infty} (C_s' p_{2n+1}^s + D_s' q_{2n+1}^s) \Big] \; (n \geqq 0), \\
-\overline{D}_{2n+1} &= -\overline{D}_{2n+1}^{(0)} - \Big[\sum_{s=0}^{\infty} (\overline{C}_{2s+1} j_{2n+1}^{2s+1} - \overline{D}_{2s+1} k_{2n+1}^{2s+1}) + \\
&\qquad + \sum_{s=1}^{\infty} (C_s' r_{2n+1}^s + D_s' t_{2n+1}^s) \Big] \; (n \geqq 1), \\
C_n' &= C_n'^{(0)} - \Big[\sum_{s=0}^{\infty} (\overline{C}_{2s+1} h_n'^{2s+1} - \overline{D}_{2s+1} i_n'^{2s+1}) - \\
&\qquad - (C_n' p_n' + D_n' q_n') \Big] \; (n \geqq 1), \\
D_n' &= D_n'^{(0)} - \Big[\sum_{s=0}^{\infty} (\overline{C}_{2s+1} j_n'^{2s+1} - \overline{D}_{2s+1} k_n'^{2s+1}) - \\
&\qquad - (C_n' r_n' + D_n' t_n') \Big] \; (n \geqq 1),
\end{aligned}\right\} \quad (10)$$

which is the analogon to the system (10) of section (7).

From the formulae

$$\sigma_y = p' \frac{z}{c}, \; \sigma_z = 0, \; \tau_{yz} = 0 \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

(describing the state of stress of the unperforated strip) it follows

$$C_n'^{(0)} = 0 \qquad D_n'^{(0)} = 0. \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

As to $\sigma_r^{(0)}, \tau_{r\varphi}^{(0)}$ and $\sigma_\varphi^{(0)}$, it can easily be checked, that

$$\left.\begin{aligned}\sigma_r^{(0)} &= \tfrac{1}{4}\,p'\,\mu\sin\varphi + \tfrac{1}{4}\,p'\,\mu\sin 3\varphi,\\ \tau_{r\varphi}^{(0)} &= -\tfrac{1}{4}\,p'\,\mu\cos\varphi + \tfrac{1}{4}\,p'\,\mu\cos 3\varphi,\\ \sigma_\varphi^{(0)} &= \tfrac{3}{4}\,p'\,\mu\sin\varphi - \tfrac{1}{4}\,p'\,\mu\sin 3\varphi,\end{aligned}\right\}\ \text{with } \mu = a/c. \quad . \quad . \quad (13)$$

Comparison of (8) with (13) leads to

$$\left.\begin{aligned}\overline{C}_1^{(0)} = -\overline{D}_1^{(0)} = -\tfrac{1}{4}\,p'\,\mu,\ \overline{C}_3^{(0)} = -\tfrac{1}{4}\,p'\,\mu,\ \overline{D}_3^{(0)} = -\tfrac{1}{4}\,p'\,\mu\\ \overline{C}_{2n+1}^{(0)} = \overline{D}_{2n+1}^{(0)} = 0.\end{aligned}\right\} \quad . \quad (14)$$

For the iterative process by which the system (10) can be solved, we refer to art. 7 on the understanding that the symmetrical stress-functions occurring there are replaced here by anti-symmetrical ones. Analytically the iteration is described by the set of formulae

$$\left.\begin{aligned}
\overline{C}_{2n+1}^{(i+1)} &= -\Big[\sum_{s=0}^{\infty}\big(\overline{C}_{2s+1}^{(i)}\,h_{2n+1}^{2s+1} - \overline{D}_{2s+1}^{(i)}\,i_{2n+1}^{2s+1}\big) + \\
&\qquad + \sum_{s=1}^{\infty}\big(C_s'^{(i)}\,p_{2n+1}^{s} + D_s'^{(i)}\,q_{2n+1}^{s}\big)\Big] \quad (n \geqq 0),\\
-\overline{D}_{2n+1}^{(i+1)} &= -\Big[\sum_{s=0}^{\infty}\big(\overline{C}_{2s+1}^{(i)}\,j_{2n+1}^{2s+1} - \overline{D}_{2s+1}^{(i)}\,k_{2n+1}^{2s+1}\big) + \\
&\qquad + \sum_{s=1}^{\infty}\big(C_s'^{(i)}\,r_{2n+1}^{s} + D_s'^{(i)}\,t_{2n+1}^{s}\big)\Big] \quad (n \geqq 1),\\
C_n'^{(i+1)} &= -\Big[\sum_{s=0}^{\infty}\big(\overline{C}_{2s+1}^{(i)}\,h_n'^{2s+1} - \overline{D}_{2s+1}^{(i)}\,i_n'^{2s+1}\big) - \\
&\qquad - \big(C_n'^{(i)}\,p_n' + D_n'^{(i)}\,q_n'\big)\Big] \quad (n \geqq 1),\\
D_n'^{(i+1)} &= -\Big[\sum_{s=0}^{\infty}\big(\overline{C}_{2s+1}^{(i)}\,j_n'^{2s+1} - \overline{D}_{2s+1}^{(i)}\,k_n'^{2s+1}\big) - \\
&\qquad - \big(C_n'^{(i)}\,r_n' + D_n'^{(i)}\,t_n'\big)\Big] \quad (n \geqq 1),
\end{aligned}\right\} \quad (15)$$

by which in succession

$$\overline{C}_{2n+1}^{(0)},\ \overline{D}_{2n+1}^{(0)},\ C_n'^{(0)},\ D_n'^{(0)},\ \overline{C}_{2n+1}^{(1)},\ \overline{D}_{2n+1}^{(1)},\ C_n'^{(1)},\ D_n'^{(1)}\ \text{a.s.o.}$$

can be computed. The convergence of the process being taken for granted, the solution of (10) is given by

$$\left.\begin{aligned}\overline{C}_{2n+1} = \sum_{i=0}^{\infty} C_{2n+1}^{(i)}\ (n \geqq 0),\ \overline{D}_{2n+1} = \sum_{i=0}^{\infty} D_{2n+1}^{(i)}\ (n \geqq 1),\\ C_n' = \sum_{i=0}^{\infty} C_n'^{(i)}\ (n \geqq 1),\ D_n' = \sum_{i=0}^{\infty} D_n'^{(i)}\ (n \geqq 1).\end{aligned}\right\} \quad (16)$$

The stresses can now be calculated in the same way as in art. 7. Particularly it may be stated that the tangential normal stresses along the boundaries I, II and III can be represented by simple formulae, because of the practicability of the double summations. By summing up σ_{φ} (resp. $\sigma_{\varphi}^{(0)}$) from (2), (7) and (13), and σ_y (resp. $\sigma_y^{(0)}$) from (3), (5), (6) and (12), we find — with due regard to (10), (12) and (14) —:

$$\left.\begin{aligned} \sigma_{\varphi} &= 4 \sum_{n=0}^{\infty} \overline{D}_{2n+1} \sin(2n+1)\varphi && \text{along the boundaries I} \\ \sigma_y &= \pm p' \pm 4 \sum_{n=1}^{\infty} D'_n \cos 2\pi n \frac{y}{b} && \text{along the boundaries I and II.} \end{aligned}\right\} \quad (17)$$

After what has been remarked in art. 7 it seems unnecessary to enter upon the stress-calculation of an arbitrary point of the stressfield and consequently the theoretical treatment of the subject could be considered as closed, if not one difficulty remained to be faced which also was encountered in the treatment of the strip subject to tension. In the same way, as in art. 7 it proved itself to be necessary to relate the tensional stress p' to the total normal load of the strip, we have now to express the bending stress p' in terms of the total bending load M. With the narrowest section of the strip in view we find

$$\left.\begin{aligned} M = 2\int_a^c \sigma_y z_{\mathrm{I}}\, dz_{\mathrm{I}} + 2\int_a^c \sigma_{y\mathrm{I}} z_{\mathrm{I}}\, dz_{\mathrm{I}} + & \\ + 2\int_{-2c}^{0} \sigma_{y\mathrm{II}} (z_{\mathrm{II}} + c)\, dz_{\mathrm{II}} - 2 \int_{-(c+a)}^{-(c-a)} \sigma_{y\mathrm{II}} (z_{\mathrm{II}} + c)\, dz_{\mathrm{II}}, & \end{aligned}\right\} \quad (18)$$

in which formula the notation is in agreement with (7, 21) and σ_y is represented by (11). It recommends itself to introduce the maximum bending stress p, occurring in the unperforated strip of width $2c$, which is given by

$$M = {}^2/_3\, p c^2. \quad \ldots\ldots\ldots \quad (19)$$

Then (18) passes into

$$\left.\begin{aligned} \frac{p}{p'} = 1 - \mu^3 + \frac{3}{c^2 p'} \Bigg[\int_a^c \sigma_{y\mathrm{I}} z_{\mathrm{I}}\, dz_{\mathrm{I}} + & \\ + \int_{-2c}^{0} \sigma_{y\mathrm{II}} (z_{\mathrm{II}} + c)\, dz_{\mathrm{II}} - \int_{-(c+a)}^{-(c-a)} \sigma_{y\mathrm{II}} (z_{\mathrm{II}} + c)\, dz_{\mathrm{II}} \Bigg]. & \end{aligned}\right\} \quad (20)$$

The stress $\sigma_{y\mathrm{I}}$, occurring in the first right-hand integral is found by multiplying the stresses σ_r (4, 16) by the corresponding constants occurring in

(1) and summing them up for $\eta = 0$. The integral itself is split up in the same way as the integral (7, 23)

$$\int_a^c \sigma_{y\,\mathrm{I}}\, z_\mathrm{I}\, dz_\mathrm{I} = I_c - I_a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

and, with the abbreviations (4, 19) it is found that

$$\left.\begin{aligned} I_c = \frac{b^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Bigg[& \sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, h_n'^{2s+1} - \bar{D}_{2s+1}\, i_n'^{2s+1}) + \\ & + 2\pi n \frac{\lambda}{\mu} \sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, j_n'^{2s+1} - D_{2s+1}\, k_n'^{2s+1}) \Bigg]. \end{aligned}\right\} \qquad (22)$$

The answer obtained for I_a is transformed firstly by identifying the series

$$U_{2s+1},\ \frac{\partial \bar{U}_{2s+1}}{\partial z},\ \bar{U}^*_{2s+1},\ \frac{\partial \bar{U}^*_{2s+1}}{\partial z}$$

(4, 10) and (4, 12) for the point ($y = 0$, $z = a$) with the series (3, 19) for the point $r = z = a$, $\varphi = \frac{1}{2}\pi$) and secondly by using the abbreviations (3, 19) and (3, 17). The result is

$$\left.\begin{aligned} I_a = & -\tfrac{1}{2} a^2 \pi \lambda (C_1 + \tfrac{1}{2} \bar{C}_3 + \tfrac{1}{2} \bar{D}_3) - \\ & - a^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[\bar{D}_{2n+1} - \sum_{s=0}^{\infty} (\bar{C}_{2s+1}\, j_{2n+1}^{2s+1} - \bar{D}_{2s+1}\, k_{2n+1}^{2s+1}) \right] \end{aligned}\right\} \qquad (23)$$

The second right-hand integral of (20), in which $\sigma_{y\,\mathrm{II}}$ must be replaced by (5, 6) becomes, by the use of the abbreviations (5, 8)

$$\left.\begin{aligned} \int_{-2c}^{0} \sigma_{y\,\mathrm{II}} (z_\mathrm{II} + c)\, dz_\mathrm{II} = \frac{b^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Big\{ & C_n' - (C_n'\, p_n' + D_n'\, q_n') + \\ & + 2\pi n \frac{\lambda}{\mu} [D_n' - (C_n'\, r_n' + D_n'\, t_n')] \Big\}. \end{aligned}\right\} \qquad (24)$$

The third integral has to be transformed into the coordinate z_I of the system of coordinates corresponding to the holes of the strip, by substituting $z_\mathrm{II} = z_\mathrm{I} - c$. The new integral takes the form $\int_{-a}^{+a} \sigma_{y\,\mathrm{II}}\, z_\mathrm{I}\, dz_\mathrm{I}$, in which σ can be derived from the series (6, 7) and (6, 8) by putting $r = z$ and $\varphi = \frac{1}{2}\pi$. Evidently the even powers of z do not contribute to the integral and — using the notations (6, 10) — we have

$$\int_{-(c+a)}^{-(c-a)} \sigma_{y\,\mathrm{II}} (z_\mathrm{II} + c)\, dz_\mathrm{II} = a^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{s=1}^{\infty} (C_s'\, r_{2n+1}^s + D_s'\, t_{2n+1}^s). \qquad (25)$$

TABLE 1.

The coefficients $\overline{C}_{2n+1}$, $\overline{D}_{2n+1}$, C'_n, D'_n with reference to p'.

	$\mu=0.20$			$\mu=0.40$			$\mu=0.60$	
	$\lambda=0.10$	$\lambda=0.20$	$\lambda=0.30$	$\lambda=0.10$	$\lambda=0.20$	$\lambda=0.30$	$\lambda=0.15$	$\lambda=0.30$
$\overline{C}_1/p'$	-0.04993	-0.04950	-0.04785	-0.09890	-0.09754	-0.09511	-0.14192	-0.13327
$\overline{C}_3/p'$	-0.05007	-0.04773	-0.04074	-0.10520	-0.10163	-0.08434	-0.18568	-0.15662
$\overline{C}_5/p'$	$+0.00001^{9}$	$+0.00040$	$+0.00319$	$+0.00135$	$+0.00212$	$+0.00790$	$+0.01917$	$+0.02731$
$\overline{C}_7/p'$	—	$+0.00004^{0}$	$+0.00068$	-0.00015	-0.00004	$+0.00121$	-0.00439	-0.00108
$\overline{C}_9/p'$	—	$+0.00000^{4}$	$+0.00011$	$+0.00001^{3}$	$+0.00001^{9}$	$+0.00024$	$+0.00075$	$+0.00104$
$\overline{C}_{11}/p'$	—	—	$+0.00001^{6}$	—	—	$+0.00003^{1}$	-0.00011	-0.00006
$\overline{C}_{13}/p'$	—	—	—	—	—	—	$+0.00002^{2}$	$+0.00001^{6}$
$\overline{D}_1/p'$	$+0.04993$	$+0.04950$	$+0.04785$	$+0.09890$	$+0.09754$	$+0.09511$	$+0.14192$	$+0.13327$
$\overline{D}_3/p'$	-0.05007	-0.04785	-0.04194	-0.10453	-0.10106	-0.08626	-0.17488	-0.14919
$\overline{D}_5/p'$	$+0.00001^{9}$	$+0.00039$	$+0.00298$	$+0.00126$	$+0.00200$	$+0.00735$	$+0.01586$	$+0.02357$
$\overline{D}_7/p'$	—	$+0.00003^{9}$	$+0.00064$	-0.00015	-0.00004	$+0.00116$	-0.00369	-0.00063
$\overline{D}_9/p'$	—	$+0.00000^{4}$	$+0.00011$	$+0.00001^{2}$	$+0.00001^{8}$	$+0.00023$	$+0.00063$	$+0.00092$
$\overline{D}_{11}/p'$	—	—	$+0.00001^{6}$	—	—	$+0.00003^{0}$	-0.00009	-0.00005
$\overline{D}_{13}/p'$	—	—	—	—	—	—	$+0.00001^{8}$	$+0.00001^{5}$
C'_1/p'	-0.00114	-0.00089	-0.00018	-0.00313	-0.01685	-0.01612	-0.01797	-0.07576
C'_2/p'	-0.00047	-0.00001^{4}	—	-0.00876	-0.00645	-0.00122	-0.04513	-0.02808
C'_3/p'	-0.00007^{1}	—	—	-0.00666	-0.00091	-0.00003^{7}	-1.03250	-0.00403
C'_4/p'	-0.00000^{7}	—	—	-0.00333	-0.00009	—	-0.01563	-0.00045
C'_5/p'	—	—	—	-0.00133	-0.00000^{7}	—	-0.00610	-0.00003^{1}
C'_6/p'	—	—	—	-0.00047	—	—	-0.00210	—
C'_7/p'	—	—	—	-0.00015	—	—	-0.00067	—
C'_8/p'	—	—	—	-0.00004	—	—	-0.00021	—
C'_9/p'	—	—	—	-0.00001^{3}	—	—	-0.00006^{4}	—
C'_{10}/p'	—	—	—	-0.00000^{4}	—	—	-0.00002^{0}	—
D'_1/p'	-0.00060	-0.00071	-0.00015	$+0.00232$	-0.00810	-0.01085	$+0.01381$	-0.02912
D'_2/p'	-0.00038	-0.00001^{3}	—	-0.00422	-0.00493	-0.00102	-0.01814	-0.01889
D'_3/p'	-0.00006^{2}	—	—	-0.00455	-0.00076	-0.00003^{2}	-0.01946	-0.00295
D'_4/p'	-0.00000^{7}	—	—	-0.00255	-0.00008	—	-0.01054	-0.00029
D'_5/p'	—	—	—	-0.00108	-0.00000^{6}	—	-0.00432	-0.00002^{2}
D'_6/p'	—	—	—	-0.00040	—	—	-0.00153	—
D'_7/p'	—	—	—	-0.00013	—	—	-0.00049	—
D'_8/p'	—	—	—	-0.00004	—	—	-0.00015	—
D'_9/p'	—	—	—	-0.00001^{1}	—	—	-0.00004^{6}	—
D'_{10}/p'	—	—	—	-0.00000^{3}	—	—	-0.00001^{4}	—

Herewith all required data for the computation of p/p' (20) are available and the result is

$$\frac{p}{p'} = 1 - \mu^3 + \frac{3}{c^2 p'} \left[\tfrac{1}{2} a^2 \pi \lambda (\bar{C}_1 + \tfrac{1}{2} \bar{C}_3 + \tfrac{1}{2} \bar{D}_3) + \right.$$

$$\left. + a^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} D_{2n+1}^{(0)} + \frac{b^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(C_n'^{(0)} + 2\pi n \frac{\lambda}{\mu} D_n'^{(0)}\right) \right].$$

or, with regard to (12) and (14)

$$\frac{p}{p'} = 1 + \tfrac{3}{2} \pi \lambda \mu^2 \frac{\bar{C}_1 + \tfrac{1}{2} \bar{C}_3 + \tfrac{1}{2} D_3}{p'}, \quad \ldots \ldots \quad (26)$$

from which the required "quantity of comparison" p can be calculated. For all finite values of λ and μ the ratio p/p' appears to be < 1, the constants $\bar{C}_1$, $\bar{C}_3$, $\bar{D}_3$ being negative.

All stresses, previously found in this art. can now be expressed in terms of p and consequently all data, wanted for the numerical treatment of our problem, are available. The computation of the constants $\bar{C}_{2n+1}$, $\bar{D}_{2n+1}$, C'_n, D'_n (15) and (16) needs no explanation because the analogous calculation for the strip under tension has been discussed at full length in art. *7b*. It may however be stated that the convergence of the process in this case is considerably better. The results are given in table 1.

Table 2 contains the values of the ratio p'/p (the reciprocal of (26)).

TABLE 2. The multiplicator p'/p.

λ \ μ	0	0.20	0.40	0.60
0	1.00000	1.00000	1.00000	1.00000
0.10	1.00000	1.00184	1.01560	1.08932*)
0.20	1.00000	1.00368	1.03092	
0.30	1.00000	1.00508	1.04254	1.17046

*) This value corresponds with $\lambda = 0.15$.

It will be seen that these values are much smaller than the corresponding ones of table 2 in art. *7b*. The boundary-stresses σ_φ and $\sigma_{y\mathrm{II}}$ along the circular boundary I and the straight boundary II are to be found in table 3 and 4. They are calculated with the aid of (17) and table 2. It may be left to the reader to draw graphs of σ_φ and $\sigma_{y\mathrm{II}}$, and to state that the absolute maximum of σ_φ occurs in the points $\varphi = \pm \frac{\pi}{2}$. Its value amounts to about two times the stress in the corresponding point of the unperforated strip, so that the "stress concentration factor" is nearly equal to 2. With the strip under tension the same factor amounted to 3. Furthermore it may

be stated, that $\sigma_{\varphi max}$ increases with increasing μ and decreases with increasing λ (lee-effect). But here too the effects are less pronounced than with the strip under tension.

TABLE 3. The tangential stress σ_φ along the boundary I.

σ_φ/p	$\mu=0.20$			$\mu=0.40$			$\mu=0.60$	
φ	$\lambda=0.10$	$\lambda=0.20$	$\lambda=0.30$	$\lambda=0.10$	$\lambda=0.20$	$\lambda=0.30$	$\lambda=0.15$	$\lambda=0.30$
0°	0	0	0	0	0	0	0	0
10°	−0.0655	−0.0602	−0.0388	−0.1392	−0.1323	−0.0819	−0.2333	−0.1550
20°	−0.1053	−0.0968	−0.0668	−0.2257	−0.2153	−0.1427	−0.3905	−0.2847
30°	−0.1006	−0 0921	−0.0682	−0.2210	−0.2115	−0.1495	−0.4127	−0.3340
40°	−0.0452	−0.0393	−0.0289	−0.1107	−0.1050	−0.0717	−0.2706	−0.2389
50°	+0.0529	+0.0547	+0.0518	+0.0908	+0.0921	+0.0953	+0.0333	+0.0298
60°	+0.1732	+0.1709	+0.1584	+0.3430	+0.3410	+0.3210	+0.4621	+0.4425
70°	+0.2883	+0.2826	+0.2646	+0.5885	+0.5847	+0.5501	+0.9347	+0.9097
80°	+0.3709	+0.3630	+0.3423	+0.7669	+0.7623	+0.7202	+1.3186	+1.2912
90°	+0.4008	+0.3922	+0.3707	+0.8322	+0.8274	+0.7830	+1.4687	+1.4402

TABLE 4. The normal stress σ_y along the straight boundary II.

σ_y/p	$\mu=0.20$			$\mu=0.40$			$\mu=0.60$	
y/b	$\lambda=0.10$	$\lambda=0.20$	$\lambda=0.30$	$\lambda=0.10$	$\lambda=0.20$	$\lambda=0.30$	$\lambda=0.15$	$\lambda=0.30$
0.0	+0.9977	+1.0008	+1.0045	+0.9723	+0.9737	+0.9929	+0.9113	+0.9304
0.1	+0.9996	+1.0014	+1.0046	+1.0378	+0.9989	+1.0047	+1.2013	+1.0383
0.2	+1.0026	+1.0028	+1.1049	+1.0398	+1.0395	+1.0321	+1.2075	+1.2105
0.3	+1.0036	+1.0046	+1.0053	+1.0120	+1.0551	+1 0598	+1.0675	+1.2727
0.4	+1.0033	+1.0060	+1.0056	+1.0021	+1.0509	+1.0778	+1.0130	+1.2502
0.5	+1.0030	+1.0065	+1.0057	+1.0004	+1.0468	+1.0837	+1.1028	+1.2309

The fluctuations in the boundary-stress σ_y are qualitatively the same as those of σ_y occurring in the strip under tension but the deviations from the approximative value p are much smaller here.

The maximum stress occurring in the strip as a whole is for values of $\mu <$ about 0,50 identical with the maximum stress of the straight boundaries; for values of $\mu > 0{,}50$ identical with the stress occurring in the point $\varphi = 90°$ of the circular holes, and consequently it can be said that the material of the strip is exploited at its best if $\mu \sim 0{,}50$. However it must

be emphasized that under all circumstances the maximum stress is greater than p, so that if an unperforated strip should be replaced by a perforated one with the same maximum stress, the width should be somewhat enlarged. Naturally the saving of material, which under circumstances may be the aim of introducing holes, is correspondingly reduced by this effect.

Finally the bending stiffness of the strip has to be considered. Evidently — in consequence of the symmetry of the state of stress — all cross-sections, which bisect the distance of any two consecutive holes, remain plane. The angle ψ between two successive of these cross-sections (say V_1 and V_2) obviously defines the curvature of the bent strip. Quite analogous to the train of thought, developed at the end of art. 7*b* we consider the strip as a part of an infinite plate, subject simultaneously to a bending load and to such stress-functions F and F', which annul the stresses along the boundaries I, II and III. A fibre between the planes V_1 and V_2 on a great distance $z = l$ from the y-axis will practically not be effected by the stress-functions F, F', and therefore will have a specific elongation $\triangle b/b = p'l/Ec$.

It follows that

$$\psi = \triangle b/l = p'b/Ec \text{ and } 1/R = \psi/b = p'/Ec = 1/E \,.\, p'/p \,.\, p/c$$

($1/R$ representing the mean curvature of the bent strip).

Or, with regard to (29)

$$\frac{1}{R} = \frac{3}{2}\frac{M}{Ec^3}\frac{p'}{p}$$

M representing the bending moment per unit of thickness of the strip.

It follows that by the perforation of the strip its curvature is magnified in the ratio p'/p, tabulated in table 2.

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9. *A row of rivet-holes in the neighbourhood of a straight boundary.* A third possibility of application of the formulae developed in art. 3—6 is given by a row of rivet-holes in the neighbourhood of a straight boundary. It is supposed that the boundary-stresses of every hole give rise to a resultant force P perpendicular to (and directed towards) the straight boundary. The reaction of these resultant forces consists of a continuous constant tension $p = P/b$ acting at infinity at the "opposite edge" of the infinite plate. The resultant force P is supposed to be exerted by a normal pressure $\sigma_r = \frac{2P}{\pi a} \sin \varphi$ acting on the upper-half $(0 \leqq \varphi \leqq 180°)$ of the circular boundary of the hole; no other stresses on this boundary are present.

Once again our starting point is the undisturbed infinite plate on which the straight boundary II as well as the circular boundaries are marked. Then in the centre of every circle a stress-function F_p is introduced (expressed in coordinates (r, φ) of the circle under consideration) which stands up for a force P in the direction $\varphi = 90°$. The stress-function, which with respect to the fixed system of rectangular coordinates (y, z) is identical with the infinity of stress-functions F_p — and which gives rise to a stress-field of the period b (the pitch of the holes) is in accordance with art. 3 denoted by U_p.

It can easily be verified (comp. if desired "A", 12, 4) that F_p is represented by

$$F_P = \frac{P}{2\pi}\left(r\varphi \cos \varphi + \frac{m-1}{2m}\, r \ln r \sin \varphi\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By introducing $x = re^{i\varphi} = y + iz$ the expression is transformed into rectangular coordinates:

$$F_P = \frac{P}{2\pi} Jm \left(x \ln x - \frac{m+1}{2m}\, iz \ln x\right)$$

or

$$F_P = \frac{P}{2\pi} Jm \left(-x + \int \ln\, x\, dx - \frac{m+1}{2m}\, iz \ln x\right).$$

In the latter of these formulae $-x$ can be suppressed since the stresses σ and τ only depend upon the second derivatives of F_p. The stress-function F_p^k, related to the hole the centre of which has the coordinates $y = kb$, $z = 0$, can be expressed in the coordinates y, z, used for the functions F_p, by replacing in the latter expression x by $(x - kb)$:

$$F_p^k = \frac{P}{2\pi} Jm \left[\int ln\,(x - kb)\,dx - \frac{m+1}{2m}\, iz\, ln\,(x - kb) \right]. \quad . \quad (2)$$

The function U_p which is equivalent with the whole system of stress-functions introduced in the centres of the infinite series of rivetholes, is therefore represented by

$$U_P = \frac{P}{2\pi} Jm \left[\int \chi(x)\,dx - \frac{m+1}{2m}\, iz\, \chi(x) \right] \quad . \quad . \quad . \quad . \quad (3)$$

where $\chi(x)$ is defined by (4, 1):

$$\chi(x) = ln \sin \pi x/b$$

Obviously it is our aim to derive from (3) the stresses occurring at the boundaries I and II, and to identify these stresses with the prescribed ones.

The stresses along the straight boundary II can easily be found by introducing for χ the expansion (4, 4). This leads to

$$U_P - \frac{Pb}{4\pi^2} \left[\mp \tfrac{1}{1} \eta^2 \mp \frac{1}{4m} \zeta^2 \pm \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 \pm \frac{m+1}{2m} n\zeta \right) e^{\mp n\zeta} \cos n\eta. \right. \quad (4)$$

with the corresponding stresses:

$$\left.\begin{aligned} \sigma_y &= \pm p \left[-\frac{1}{2m} + \sum_{n=1}^{\infty} \left(-\frac{1}{m} \pm \frac{m+1}{2m} n\zeta \right) e^{\mp n\zeta} \cos n\eta \right], \\ \sigma_z &= \mp p \left[+ \tfrac{1}{2} + \sum_{n=1}^{\infty} \left(1 \pm \frac{m+1}{2m} n\zeta \right) e^{\mp n\zeta} \cos n\eta \right], \\ \tau_{yz} &= -p \sum_{n=1}^{\infty} \left(\frac{m-1}{2m} \pm \frac{m+1}{2m} n\zeta \right) e^{\mp n\zeta} \sin n\eta, \end{aligned}\right\} \quad . \quad (5)$$

p standing for P/b, the other notations being the same as those in art. 4 (Comp. 4, 1, 2, 3). In particular it may be remembered that in every ambiguous sign the upper sign relates to positive values of z, the lower one to negative values of z. By putting $z = c$, resp. $\zeta = 2\pi\,\lambda/\mu$ $(\lambda = a/b$, $\mu = a/c)$ and by introducing the abbreviations

$$\left.\begin{aligned} C_n'^{(0)} &= p \left(1 + \frac{m+1}{4m} \cdot 4\pi n \frac{\lambda}{\mu} \right) e^{-2\pi n \frac{\lambda}{\mu}}, \\ D_n'^{(0)} &= p \left(\frac{m-1}{2m} + \frac{m+1}{4m} \cdot 4\pi n \frac{\lambda}{\mu} \right) e^{-2\pi n \frac{\lambda}{\mu}} \qquad (n \geqq 1) \quad (6) \end{aligned}\right\} . \quad (6)$$

it follows from (5) that the stresses along the straight boundary are represented by

$$\left.\begin{aligned}\sigma_y &= -\frac{p}{2m} - \sum_{n=1}^{\infty}(C_n'^{(0)} - 2D_n'^{(0)})\cos 2\pi n\frac{y}{b},\\ \sigma_z &= -\frac{p}{2} - \sum_{n=1}^{\infty} C_n'^{(0)}\cos 2\pi n\frac{y}{b},\\ \tau_{yz} &= -\sum_{n=1}^{\infty} D_n'^{(0)}\sin 2\pi n\frac{y}{b}.\end{aligned}\right\} \quad . \; . \; . \quad (7)$$

Afterwards the two latter of these stresses shall have to be neutralized, and in this connection we state already here, that as far as they depend upon the series $\sum_{n=1}^{\infty} D_0'^{(0)}$ $\cos 2\pi ny/b$ and $\sum_{n=1}^{\infty} D'^{(0)}\sin 2\pi ny/b$ — which obviously represent systems of equilibrium — this will be done by the introduction of a stress-function of the type (5, 5). As for the stress $\sigma_z = -p/2$, this will be removed by the superposition of the homogeneous state of stress $\sigma_z = +p/2$, $\sigma_y = \tau_{yz} = 0$.

But previously we pass to the computation of the stresses σ_r, σ_φ and $\tau_{r\varphi}$ along the boundary I. If in eq. (2) $ln(x-kb)$ is replaced by the series $-\sum_{n=1}^{\infty} x^n/nk^n b^n$ (which differs from $ln(x-kb)$ only by the absence of the non-essential constant $ln(-kb)$) it takes the form

$$\left.\begin{aligned}F_P^k = -\frac{Pb}{2\pi}\Bigg[\frac{m+1}{8m}\frac{1}{k^2}\left(\frac{r}{b}\right)^3\sin\varphi + \sum_{n=2}^{\infty}\frac{1}{k^{n-1}}\Bigg\{\frac{1}{n-1}\left(\frac{1}{n}-\frac{m+1}{4m}\right)+\\ +\frac{1}{n+1}\frac{m+1}{4m}\frac{1}{k^2}\frac{r^2}{b^2}\Bigg\}\left(\frac{r}{b}\right)^n\sin n\varphi\Bigg],\end{aligned}\right\} \quad (8)$$

With the aid of (1) and (8) $U_p = F_p + \sum_{k=1}^{\infty}(F_P^k + F_P^{-k})$ can be expressed in polar coordinates, and paying attention to the abbreviations (3, 12) it can easily be verified, that

$$\left.\begin{aligned}U_P = \frac{pb}{2\pi}\left(r\varphi\cos\varphi + \frac{m-1}{2m}r\,ln\,r\sin\varphi\right) - \frac{pb^2}{\pi}\Bigg[\frac{m+1}{8m}\sigma_2\left(\frac{r}{b}\right)^3\sin\varphi + \\ + \sum_{n=1}^{\infty}\Bigg\{\frac{1}{2n}\left(\frac{1}{2n+1}-\frac{m+1}{4m}\right)\sigma_{2n} + \\ + \frac{1}{2n+2}\frac{m+1}{4m}\sigma_{2n+2}\frac{r^2}{b^2}\Bigg\}\left(\frac{r}{b}\right)^{2n+1}\sin(2n+1)\varphi\Bigg].\end{aligned}\right\} \quad (9)$$

The corresponding stresses (comp. 2, 4) are only wanted for $r = a$. If we put

$$\left.\begin{aligned}
\overline{C}_1^{(0)} &= -\overline{D}_1^{(0)} = -\frac{p}{2\pi}\left(\frac{m-1}{2m}\frac{1}{\lambda} - \frac{m+1}{2m}\sigma_2\lambda\right). \\
C_{2n+1}^{(0)} &= -\frac{p}{2\pi}\left\{\left[2-(2n+1)\frac{m+1}{2m}\right]\sigma_{2n}+(2n-1)\frac{m+1}{2m}\sigma_{2n+2}\lambda^2\right\}\lambda^{2n-1}, \\
D_{2n+1}^{(0)} &= -\frac{p}{2\pi}\left\{\left[2-(2n+1)\frac{m+1}{2m}\right]\sigma_{2n}+(2n+1)\frac{m+1}{2m}\sigma_{2n-2}\lambda^2\right\}\lambda^{2n-1}
\end{aligned}\right\} \quad (10)$$

it is found that

$$\left.\begin{aligned}
\sigma_r &= -\frac{p}{\pi\lambda}\sin\varphi - \sum_{n=0}^{\infty}\overline{C}_{2n+1}^{(0)}\sin(2n+1)\varphi, \\
\sigma_\varphi &= -\frac{p}{2\pi}\frac{m-1}{m}\frac{1}{\lambda}\sin\varphi - \sum_{n=0}^{\infty}(\overline{C}_{2n+1}^{(0)} - 2\overline{D}_{2n+1}^{(0)})\sin(2n+1)\varphi, \\
\tau_{r\varphi} &= -\sum_{n=0}^{\infty}\overline{D}_{2n+1}^{(0)}\cos(2n+1)\varphi.
\end{aligned}\right\} \quad (11)$$

The formulae (7) and (11) provide us with the required boundary-stresses, which, as far as σ_r and $\tau_{r\varphi}$ at I, and σ_z and τ_{yz} at II are concerned, have to assume prescribed values. This is attained by the introduction of equilibrium stress-systems represented respectively by a stress function F (3, 20) defined with respect to the system of coordinates connected with the circular boundary I, and a stress function F' (6, 11), defined with respect to the system of coordinates connected with the straight boundary II.

Consequently at the latter boundary we have to deal with

1⁰. the stresses (7) corresponding to the stress-function U_p,

2⁰. the stresses $\sigma_y = 0$, $\sigma_z = p/2$, $\tau_{yz} = 0$ corresponding to the homogeneous state of stress mentioned before,

3⁰. a system of stresses, which can be calculated with the aid of (4, 18), corresponding to the stress-function F (3, 20),

4⁰. the stresses (5, 1) and (5, 7) corresponding to the stress-function F' (6, 11).

The resultant stresses σ_z and τ_{yz} must be zero and therefore the following boundary conditions must be fulfilled:

$$\left.\begin{aligned}
C'_n &= C'^{(0)}_n - \Big[C_0 h'^0_n + \sum_{s=1}^{\infty}(C_{2s}h'^{2s}_n + D_{2s}i'^{2s}_n) + \\
&\qquad + \sum_{s=0}^{\infty}(\overline{C}_{2s+1}h'^{2s+1}_n - D_{2s+1}i'^{2s+1}_n)\Big], \\
D'_n &= D'^{(0)}_n - \Big[C_0 j'^0_n + \sum_{s=1}^{\infty}(C_{2s}j'^{2s}_n + D_{2s}k'^{2s}_n) + \\
&\qquad + \sum_{s=0}^{\infty}(\overline{C}_{2s+1}j'^{2s+1}_n - \overline{D}_{2s+1}k'^{2s+1}_n)\Big].
\end{aligned}\right\} (n \geqq 1) \quad (12)$$

On account of these equations the required boundary-stress σ_y can be written as

$$\sigma_y = -\frac{p}{2m} + 4\sum_{n=1}^{\infty} D'_n \cos 2\pi n \frac{y}{b}. \quad . \quad . \quad . \quad . \qquad (13)$$

Analogously at the circular boundary I we have to deal with

1⁰. the stresses (11) due to the stress-function U_p,

2⁰. the stresses

$\sigma_r = \frac{1}{4}p(1+\cos 2\varphi)$, $\sigma_\varphi = \frac{1}{4}p(1-\cos 2\varphi)$, $\tau_{r\varphi} = \frac{1}{4}$—p sin 2φ (comp. 7, 12) due to the required homogeneous state of stress,

3⁰. a system of stresses, which can be calculated with the aid of (3, 15) and (3, 16), due to the stress-function F (3, 20),

4⁰. the stresses (6, 9) due to the stress-function F' (6, 11).

The resultant stress $\tau_{r\varphi}$ must be zero along the whole boundary I, whereas the resultant radial stress σ_r must be identical with $\sigma_r = -\frac{2P}{\pi a}\sin\varphi$ in the interval $0 \leqq \varphi \leqq 180°$ resp. with $\sigma_r = 0$ in the interval $80° \leqq \varphi \leqq 360°$, which comes to the same thing as the representation

$$\sigma_r = -\frac{p}{\pi\lambda}\sin\varphi - \frac{2p}{\pi^2\lambda} + \sum_{n=1}^{\infty}\frac{4p}{(4n^2-1)\pi^2\lambda}\cos 2n\varphi. \quad . \quad . \quad . \qquad (14)$$

in the whole interval $0° = \varphi = 360°$.

The boundary conditions which arise from these requirements can, by the introduction of the constants

$$\left.\begin{aligned} &C_0^{(0)} = -p\left(\frac{1}{4}+\frac{2}{\pi^2\lambda}\right), \qquad C_2^{(0)} = +p\left(\frac{1}{4}+\frac{4}{3\pi^2\lambda}\right), \\ &C_{2n}^{(0)} = \frac{4p}{(4n^2-1)\pi^2\lambda}\ (n>1),\ D_2^{(0)} = -\frac{p}{4},\ D_{2n}^{(0)} = 0 \quad (n>1) \end{aligned}\right\} \qquad (15)$$

be written as

$$\left.\begin{aligned} C_{2n} &= C_{2n}^{(0)} - \Big[C_0 h_{2n}^0 + \sum_{s=1}^{\infty}(C_{2s} h_{2n}^{2s} + D_{2s} i_{2n}^{2s}) + \\ &\qquad + \tfrac{1}{2}\sum_{s=1}^{\infty}(C'_s p_{2n}^s + D'_s q_{2n}^s)\Big] \quad (n \geqq 0), \\ \overline{C}_{2n+1} &= \overline{C}_{2n+1}^{(0)} - \Big[\sum_{s=0}^{\infty}(\overline{C}_{2s+1} h_{2n+1}^{2s+1} - \overline{D}_{2s+1} i_{2n+1}^{2s+1}) + \\ &\qquad + \tfrac{1}{2}\sum_{s=1}^{\infty}(C'_s p_{2n+1}^s + D'_s q_{2n+1}^s)\Big] \quad (n \geqq 0), \\ D_{2n} &= D_{2n}^{(0)} - \Big[C_0 j_{2n}^0 + \sum_{s=1}^{\infty}(C_{2s} j_{2n}^{2s} + D_{2s} k_{2n}^{2s}) + \\ &\qquad + \tfrac{1}{2}\sum_{s=1}^{\infty}(C'_s r_{2n}^s + D'_s t_{2n}^s)\Big] \quad (n \geqq 1), \\ -\overline{D}_{2n+1} &= -\overline{D}_{2n+1}^{(0)} - \Big[\sum_{s=0}^{\infty}(\overline{C}_{2s+1} j_{2n+1}^{2s+1} - \overline{D}_{2s+1} k_{2n+1}^{2s+1}) + \\ &\qquad + \tfrac{1}{2}\sum_{s=1}^{\infty}(C'_s r_{2n+1}^s + D'_s t_{2n+1}^s)\Big] \quad (n \geqq 0). \end{aligned}\right\} \qquad (16)$$

The required resultant boundary stress σ_φ can, — with due regard to (14), (15) and (16) — be transformed into

$$\left.\begin{aligned}\sigma_\varphi = -\frac{p}{\pi\lambda}\left|\sin\varphi\right| - 2C_0 - 4\sum_{n=0}^{\infty} D_{2n}\cos 2n\varphi - \\ -\frac{p}{2\pi\lambda}\frac{m-1}{m}\sin\varphi + 4\sum_{n=0}^{\infty}\bar{D}_{2n+1}\sin(2n+1)\varphi\end{aligned}\right\} \qquad (17)$$

To simplify the equations (12) and (16) we introduce new symbols for the constants

$$\bar{C}_{2n+1},\ \bar{D}_{2n+1},\ \bar{C}^{(0)}_{2n+1},\ \bar{D}^{(0)}_{2n+1}$$

by putting

$$\bar{C}_{2n+1} = C_{2n+1},\ \bar{D}_{2n+1} = -D_{2n+1}\ \text{en}\ \bar{C}^{(0)}_{2n+1} = C^{(0)}_{2n+1},\ \bar{D}^{(0)}_{1n+1} = -D^{(0)}_{2n+1} \quad (18)$$

(which must not be misunderstood in this sense, that the symbols C_{2n+1}, C_{2n+1}, $C^{(0)}_{2n+1}$, $D^{(0)}_{2n+1}$ should be identical with the equally nominated constants of eq. (3, 20) which in fact do not occur in this problem). Furthermore we declare the coefficients $h^s_n, i^s_n, j^s_n, k^s_n$ to be equal to zero in all cases where $(n+s)$ should be odd. Then it can easily be seen, that the six sets of equations (12) and (16) are identical with the following four:

$$\left.\begin{aligned}
C_n &= C^{(0)}_n - \left[C_0 h^0_n + \sum_{s=1}^{\infty}(C_s h^s_n + D_s i^s_n) + \tfrac{1}{2}\sum_{s=1}^{\infty}(C'_s p^s_n + D'_s q^s_n)\right] \ (n \geqq 0), \\
D_n &= D^{(0)}_n - \left[C_0 j^0_n + \sum_{s=1}^{\infty}(C_s j^s_n + D_s k^s_n) + \tfrac{1}{2}\sum_{s=1}^{\infty}(C'_s r^s_n + D'_s t^s_n)\right] \ n \geqq 1), \\
C'_n &= C'^{(0)}_n - \left[C_0 h'^0_n + \sum_{s=1}^{\infty}(C_s j'^s_n + D_s i'^s_n)\right] \quad (n \geqq 1), \\
D'_n &= D'^{(0)}_n - \left[C_0 j'^0_n + \sum_{s=1}^{\infty}(C_s j'^s_n + D_s k'^s_n)\right] \quad (n \geqq 1)
\end{aligned}\right\} \qquad (19)$$

which can be solved in the well-known iterative way.

We restrict ourselves to the reproduction of a number of numerical results. Table 1 contains for some values of λ and μ, and $m = 3, 5$, the constants $C^{(0)}_n$, $D^{(0)}_n$, $C'^{(0)}_n$, $D'^{(0)}_n$ in terms of p (comp. (6), (10) and (15)). Table 2 gives a survey of the constants C_n, D_n, C'_n, D'_n, whereas the final results for σ_φ and σ_y at the boundaries I and II are taken up in table 3 and 4.

The stresses σ_φ at the boundary I are discontinuous in their first derivative at the points $\varphi = 0$ and $\varphi = 180°$, where the prescribed radial stress has the same property.

As a rule the stress-maximum occurs in these points, but there are cases too in which the effect of the straight boundary manifests itself in a shift of those points of maximum stress. With regard to the normal stress σ_y at the straight boundary II it can be stated that it is nearly constant and equal to $-p/2m$ for great valeus of λ/μ. Tensional stresses only occur at this boundary if $\lambda/\mu <$ about 1, and then increase rapidly as λ/μ diminishes.

TABLE I. The initial constants of the iteration: $C_n^{(0)}$, $D_n^{(0)}$, $C_n'^{(0)}$, $D_n'^{(0)}$ in terms of p.

	$C_0^{(0)}$	$C_2^{(0)}$	$C_4^{(0)}$	$C_6^{(0)}$	$C_8^{(0)}$	$C_{10}^{(0)}$	$C_{12}^{(0)}$	$C_{14}^{(0)}$	$D_2^{(0)}$
$\lambda = 0.10$	−2.27642	+1.60094	+0.27019	+0.11580	+0.06433	+0.04094	—	—	−0.25000
$\lambda = 0.20$	−1.26321	+0.92547	+0.13510	+0.05790	+0.03217	+0.02047	—	—	−0.25000
$\lambda = 0.30$	−0.92547	+0.70032	+0.09006	+0.03860	+0.02144	+0.01365	+0.00945	+0.00693	−0.25000

		$2n+1=1$	$2n-1=3$	$2n+1=5$	$2n+1=7$	$2n+1=9$	$2n+1=11$	$2n+1=13$
$\lambda = 0.10$	$C_{2n+1}^{(0)}$	−0.55156	−0.00198	+0.00021	$+0.00000^{4}$	—	—	—
	$D_{2n+1}^{(0)}$	−0.55156	+0.00220	−0.00020	-0.00000^{4}	—	—	—
$\lambda = 0.20$	$C_{2n+1}^{(0)}$	−0.25055	−0.00463	+0.00157	+0.00012	$+0.00000^{7}$	—	—
	$D_{2n+1}^{(0)}$	−0.25055	+0.00640	−0.00151	−0.00012	-0.00000^{7}	—	—
$\lambda = 0.30$	$C_{2n+1}^{(0)}$	−0.12898	−0.00860	+0.00489	+0.00087	+0.00012	$+0.00001^{4}$	$+0.00000^{2}$
	$D_{2n+1}^{(0)}$	−0.13898	+0.01458	−0.00438	−0.00083	−0.00011	-0.00001^{4}	-0.00000^{2}

		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$\frac{\lambda}{\mu} = \frac{3}{2}$	$C_n'^{(0)}$	+0.00057	—	—	—	—	—	—	—
	$D_n'^{(0)}$	+0.00052	—	—	—	—	—	—	—
$\frac{\lambda}{\mu} = 1$	$C_n'^{(0)}$	+0.00941	$+0.00003^{2}$	—	—	—	—	—	—
	$D_n'^{(0)}$	+0.00821	$+0.00002^{9}$	—	—	—	—	—	—
$\frac{\lambda}{\mu} = \frac{3}{4}$	$C_n'^{(0)}$	+0.03620	+0.00057	$+0.00000^{7}$	—	—	—	—	—
	$D_n'^{(0)}$	+0.03042	+0.00052	$+0.00000^{7}$	—	—	—	—	—
$\frac{\lambda}{\mu} = \frac{1}{2}$	$C_n'^{(0)}$	+0.13049	+0.00941	+0.00057	+0.00003	$+0.00002^{2}$	—	—	—
	$D_n'^{(0)}$	+0.10271	+0.00821	+0.00052	+0.00003	$+0.00000^{2}$	—	—	—
$\frac{\lambda}{\mu} = \frac{1}{4}$	$C_n'^{(0)}$	+0.41780	+0.13039	+0.03620	+0.00941	+0.00235	+0.00057	+0.00014	$+0.00003^{2}$
	$D_n'^{(0)}$	+0.28416	+0.10271	+0.03042	+0.00821	+0.00210	+0.00052	+0.00012	$+0.00002^{2}$

TABLE 2. The constants of the stressfunctions F en F' in terms of p.

	$\lambda=0.10$						$\lambda=0.20$						$\lambda=0.30$							
	$\mu=0.00$		$\mu=0.20$		$\mu=0.40$		$\mu=0.00$		$\mu=0.20$		$\mu=0.40$		$\mu=0.00$		$\mu=0.20$		$\mu=0.40$		$\mu=0.60$	
$2n$	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}	C_{2n}	D_{2n}
0	−2.33842	—	−2.36343	—	−2.75796	—	−1.42753	—	−1.42782	—	−1.48980	—	−1.26260	—	−1.26261	+	−1.27221	—	−1.45556	
2	−1.61629	−0.26657	+1.67200	−0.31805	+2.13386	−0.67796	+0.95392	−0.29157	+0.95454	−0.29208	+1.05285	−0.35936	+0.76266	−0.37291	+0.76266	−0.37288	+0.77523	0.38236	+0.96608	−1.43982
4	+0.26930	+0.00086	+0.26174	+0.00827	+0.18585	+0.07561	+0.12643	+0.00784	+0.12596	+0.00829	0.07064	0.05895	+0.06582	+0.01558	+0.06579	+0.01562	+0.04401	−0.03076	−0.15456	0.18550
6	+0.11587	+0.00003^{5}	+0.11590	−0.08009^{8}	+0.12170	−0.00535	+0.05669	+0.00118	−0.05674	+0.00113	+0.06046	−0.00229	−0.03088	0.00659	−0.03088	−0.00660	0.03590	0.00355	+0.07001	0.02479
8		—			+0.06393	+0.00036	+0.03206	+0.00010	+0..3206	+0.00010	+0.03191	+0.00025	−0.02002	+0.10129	−0.02003	−0.00129	0.01973	+0.00135	−0.01685	+0.00339
10		—			+0.04096	−0.00002^{1}	−0.02046	+0.00000^{7}	+0.02046	+0.00000^{7}	−0.02047	−0.00001^{3}	+0.01341	−0.00022	+0.01342	+0.00021	+0.01347	+0.00017	0.01433	0.00052
12	—	—	—	—	—	—	—	—	—	—	—	—	−0.00942	+0.00002^{6}	+0.00942	+0.00002^{5}	+0.00943	+0.00002^{3}	0.00934	+0.00005
14	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+0.00694	−0.00001^{0}
$2n+1$	$\bar{C}_{2n+1}$	$\bar{D}_{2n+1}$	C_{2n+1}	$\bar{D}_{2n+1}$	$\bar{C}_{2n+1}$	D_{2n+1}	C_{2n+1}	$\bar{D}_{2n+1}$	C_{2n+1}	D_{2n+1}	C_{2n+1}	$\bar{D}_{2n-1}$	$\bar{C}_{2n+1}$	$\bar{D}_{2n-1}$	$\bar{C}_{2n+1}$	D_{2n+1}	$\bar{C}_{2n+1}$	D_{2n+1}	C_{2n+1}	$\bar{D}_{2n+1}$
1	−0.55156	+0.55156	−0.55837	+0.55837	−0.68897	+0.68897	−0.25053	+0.25053	−0.25050	+0.25060	−0.27727	+0.27727	−0.13883	+0.13883	−0.13883	+0.13883	−0.14330	0.14330	0.22566	+0.22566
3	−0.00161	−0.00183	+0.02816	+0.02701	+0.26983	+0.23755	−0.00199	−0.00376	−0.00115	−0.00296	+0.10336	+0.08784	−0.00090	−0.00695	−0.00089	−0.00693	−0.02077	−0.01108	+0.26892	+0.17456
5	+0.00022	+0.00021	−0.00097	−0.00096	−0.02180	−0.01958	+0.00191	+0.00185	+0.00174	+0.00168	−0.01580	−0.01458	+0.00700	+0.00649	+0.00699	+0.00648	−0.00432	−0.00402	−0.10831	0.08793
7	—	—	—	—	+0 00159	+0.00142	+0.00015	+0.00015	+0.00016	+0.10016	+0.00082	+0.00075	+0.00123	−0.00120	+0.00123	+0.00120	+0.00226	−0.00217	+0.01072	+0.00798
9	—	—	—	—	−0.00009^{6}	−0.00008^{7}	+0.00000^{9}	+0.00000^{9}	+0.00000^{9}	−0.00000^{9}	−0.00003^{2}	−0.00002^{8}	−0.00017	+0.00016	+0.00017	+0.00016	−0.00011	−0.00010	−0.00171	0.00128
11	—	—	—	—	—	—	—	—	—	—	—	—	+0.00001^{9}	+0.00001^{9}	+0.00001^{9}	+0.00001^{9}	+0.00001^{8}	+0.00001^{8}	+0.00027	+0.00023
13	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	−0.00004^{0}	−0.00003^{7}
n	C'_n	D'_n	C'_n	D'_n	C'_u	D'_n	C'_n	D'_n	C'_n	D'_n	C'_n	D'_n	C'_n	D'_n	C_n	D_n	C_n	D_n	C_n	D_n
1	—	—	+0.27806	+0.21630	+1.01836	+0.64143	—	—	+0.03553	+0.03075	+0.53435	+0.40197	—	—	+0.00370	+0.00333	+0.22915	+0.18552	+1.05930	+0.74026
2	—	—	+0.03082	+0.02670	+0.53359	+0.39976	—	—	+0.00019^{0}	+0.00017^{5}	+0.06880	+0.05766	—	—	—	—	+0.00588	+0.00512	+0.19670	+0.14687
3	—	—	+0.00246	+0.00223	+0.20815	+0.16782	—	—	—	—	+0.00600	+0.00523	—	—	—	—	+0.00010^{9}	+0.00009^{6}	+0.02619	+0.02022
4	—	—	+0.00016^{7}	+0.00015^{5}	+0.06935	+0.05813	—	—	—	—	+0.00044	+0.00041	—	—	—	—	—	—	+0.00297	+0.00232
5	—	—	—	—	+00.2101	+0.01801	—	—	—	—	+0.00003^{3}	+0.00003^{2}	—	—	—	—	—	—	+0.00029^{8}	+0.00023^{5}
6	—	—	—	—	+0.00597	+0.00520	—	—	—	—	—	—	—	—	—	—	—	—	+0.00002^{7}	+0 00002^{2}
7	—	—	—	—	+0.00165	+0.00142	—	—	—	—	—	—	—	—	—	—	—	—	—	
8	—	—	—	—	+0.00044	+0.00038	—	—	—	—	—	—	—	—	—	—	—	—	—	
9	—	—	—	—	+0.00012^{0}	+0.00010^{6}	—	—	—	—	—	—	—	—	—	—	—	—		
10	—	—	—	—	+0.00003^{3}	+0.00003^{0}	—	—	—	—	—	—	—	—	—	—	—	—		

TABLE 3. The tangentialstress σ_φ at the circular boundary I.

σ_φ/p	$\lambda=0.10$			$\lambda=0.20$			$\lambda=0.30$			
φ	$\mu=0.00$	$\mu=0.20$	$\mu=0.40$	$\mu=0.00$	$\mu=0.20$	$\mu=0.40$	$\mu=0.00$	$\mu=0.30$	$\mu=0.40$	$\mu=0.60$
+90°	+1.5019	+1.2229	−0.1199	+0.5253	+0.5187	−0.1675	+0.1580	+0.1580	−0.0342	−1.4239
+80°	+1.5978	+1.3551	+0.3096	+0.6143	+0.6090	+0.0658	+0.2559	+0.2558	+0.1137	−0.8203
+70°	+1.8768	+1.7328	+1.4716	+0.8728	+0.8705	+0.6924	+0.5438	+0.5439	+0.5211	+0.6546
+60°	+2.3142	+2.3041	+3.0515	+1.2747	+1 2760	+1.5315	+1.0026	+1.0027	+1.1006	+2.2996
+50°	+2.8712	+2.9966	+4.6880	+1.7782	+1.7823	+2.3802	+1.5921	+1.5924	+1.7560	+3.5276
+40°	+3.5001	+3.7324	+6.0981	+2.3296	+2.3344	+3.0912	+2.2452	+2.2454	+2.4065	+4.1342
+30°	+4.1481	+4.4420	+7.1290	+2.8684	+2.8735	+3.6030	+2.8704	+2.8705	+2.9854	+4.2391
+20°	+4.7647	+5.0717	+7.7458	+3.3446	+3.3484	+3.9255	+3.3779	+3.3778	+3.4393	+4.1042
+10°	+5.3075	+5.5860	+7.9915	+3.7223	+3.7246	+4.1054	+3.7229	+3.7226	+3.7474	+3.9671
0°	+5.7396	+5.9664	+7.9454	+3.9848	+3.9859	+4.1894	+3.9221	+3.9218	+3.9303	+3.9226
−10°	+4.9421	+5.1029	+6.5890	+3.5741	+3.5742	+3.6502	+3.6392	+3.6389	+3.6430	+3.5746
−20°	+4.0441	+4.1419	+5.1395	+3.0584	+3.0580	+3.0581	+3.2482	+3.2480	+3.2502	+3.1892
−30°	+3.0925	+3.1331	+3.6928	+2.4576	+2 4572	+2.4207	+2.7298	+2.7298	+2.7282	+2.6934
−40°	+2.1385	+2.1328	+2 3284	+1.8026	+1.8028	+1.7536	+2.0936	+2.0937	+2.0860	+2.0721
−50°	+1.2416	+1.2011	+1.1128	+1.1424	+1.1420	+1.0922	+1.3989	+1.3990	+1.3860	+1.3873
−60°	+0.4632	+0.3980	+0.1017	+0.5351	+0.5349	+0.4888	+0.7339	+0.7339	+0.7214	+0.7299
−70°	−0.1404	−0.2215	−0.6576	+0.0446	+0.0444	+0.0026	+0.1873	+0.1875	+0.1822	+0.1858
−80°	−0.5226	−0.6129	−1.1290	−0.2745	−0.2745	−0.3135	−0.1698	−0.1697	−0.1656	+0.1688
−90°	−0.6535	−0.7467	−1.2889	−0.3851	−0.3852	−0.4232	−0.2938	−0.2936	+0.2853	−0.2952

TABLE 4. The stress σ_y at the straight boundary II.

σ_y/p	$\lambda=0.10$		$\lambda=0.20$		$\lambda=0.30$		
y/b	$\mu=0.20$	$\mu=0.40$	$\mu=0.20$	$\mu=1.40$	$\mu=0.20$	$\mu=0.40$	$\mu=0.60$
0.0	+0.8387	+5.0263	−0.0192	+1.7184	−0.1295	+0.6201	+3.4969
0.1	+0.5868	+1.9418	−0.0431	+2 2213	−0.1321	+0.4637	+2.4038
0.2	+0.9311	−1.0420	−0.1054	+0.1511	−0.1387	+0.0696	+0.2272
0.3	−0.4893	−1.6767	−0.1814	−0.8094	−0.1470	−0.3884	−1.4739
0.4	−0.8075	−1.6479	−0.2422	−1.3671	−0.1536	−0.7368	−2.3354
0.5	−0.9096	−1.6040	−0.2652	−1.5395	−0.1562	−0.8648	−2.5889

Mechanics. — *Over de spanningstoestanden in prismatische assen met door twee paar orthogonale cirkelbogen begrensde doorsneden.* By C. DE BEER. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of October 27, 1945.)

1. *Inleiding.*

Verschillende schrijvers [1]) hebben reeds min of meer uitgebreid onderzocht welke invloed door het maken van één orthogonaal-cirkelvormige „insnijding" in een cirkelcylindrische as wordt uitgeoefend op de maximale schuifspanning, optredend tengevolge van een wringbelasting. Het lag voor de hand het minstens even belangrijke probleem van een as, voorzien van twee dergelijke, diametraal geplaatste insnijdingen ook tot oplossing te brengen. De grootheden betrekking hebbende op buiging, trek en druk kunnen wegens de symmetrische gedaante van de doorsnede (fig. 1) eenvoudig berekend worden, en zijn volledigheidshalve opgenomen. Minder eenvoudig is de berekening van de grootheden waarmee de wringingsproblemen worden opgelost. Door den parameter φ, die in fig. 1 is aangegeven, te laten varieeren tusschen 0 en $\pi/2$ wordt het geheele stelsel assen, gelegen tusschen den vollen cirkelcylinder met twee oneindig kleine half-cirkelvormige insnijdingen en de dunne strip met bi-concave doorsnede, achtereenvolgens doorloopen. Het is duidelijk dat voor kleine waarden van φ bij een gewrongen as van een spanningsconcentratie voor de schuifspanningen gesproken moet worden, die zooals bekend in het limietgeval $\varphi = 0$ de waarde 2 bereikt. Te verwachten is ook dat de twee insnijdingen elkaar bij een kleine waarde van φ niet zullen beïnvloeden, zoodat voor deze gevallen een vergelijking met de bestaande literatuur (zie noot 1) mogelijk is.

2. *Grondslagen der berekening.*

Zooals bekend is [2]) kunnen bij een gewrongen prismatische as de schuifspanningen in een normaaldoorsnede bij gebruik van een Cartesisch coördinatenstelsel, waarvan de x-as evenwijdig aan de beschrijvenden is, worden voorgesteld door de volgende betrekkingen:

$$\tau_{xz} = -\frac{\partial \Phi}{\partial y} \qquad \tau_{xy} = +\frac{\partial \Phi}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

[1]) T. H. GRONWALL: On the influence of keyways on the stress-distribution in cylindrical shafts. Trans. of the American Mathem. Soc.: Vol. 20.

L. A. WIGGLESWORTH and A. C. STEVENSON: Flexure and torsion of cylinders with cross-sections bounded by orthogonal arcs: Proc. of the Royal Soc. of London, Series A. Vol. 170 (1939).

W. M. SHEPHERD: The torsion and flexure of shafting with keyways or cracks: Proc. of the Royal Soc. of London, Series A, Vol. 138 (1932).

A. VAN WIJNGAARDEN: Enige toepassingen van FOURIER-integralen op elastische problemen. Diss. 1945 (Meinema, Delft).

[2]) Zie b.v.: BIEZENO–GRAMMEL, Technische Dynamik, Springer 1939, blz. 112 e.v.

waarin Φ een oplossing van de differentiaalvergelijking

$$\triangle' \Phi = -2\,G\omega \qquad \left(\triangle' \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \quad . \; . \; . \; . \; . \; (2)$$

is, die de randvoorwaarden van het probleem bevredigt. In (2) is G de glijdingsmodulus en ω de specifieke wringingshoek. De randvoorwaarden worden geformuleerd door den eisch dat in ieder punt van het doorsnede-contour de functie Φ de waarde nul aan moet nemen.

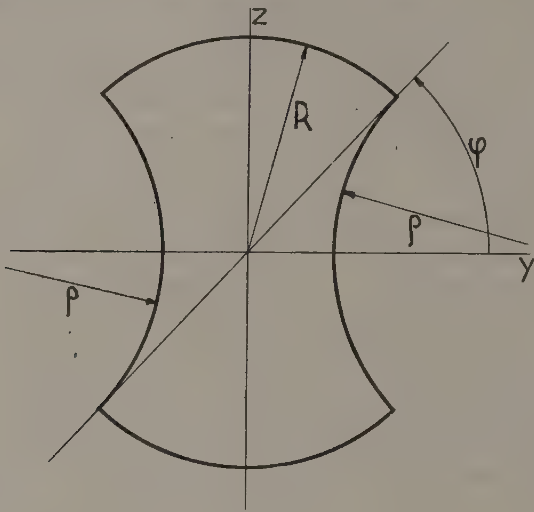

Fig. 1.

Zooals in fig. 1 aangegeven is bestaat het contour uit bogen van drie verschillende cirkels, waarvan één den straal R, de beide andere den straal ϱ bezitten. In het volgende zullen wij spreken van den cirkel R en de cirkels ϱ en bedoelen dan de bogen van de cirkels met stralen R resp. ϱ die den omtrek der doorsnede vormen.

Een vaak toegepast hulpmiddel om een probleem als het bovenstaande tot oplossing te brengen, nl. het splitsen van de functie Φ in twee hulpfuncties Φ^* en Φ^{**} zullen we ook hier toepassen. Daartoe stellen wij:

$$\Phi = \Phi^* + \Phi^{**}$$

en bepalen Φ^* en Φ^{**} zoodanig, dat

$$\begin{array}{llll} a. & \triangle' \Phi^* = -2\,G\omega, & \Phi^*_R = 0 & \Phi^*_\varrho \neq 0 \\ b. & \triangle' \Phi^{**} = 0 & , \; \Phi^{**}_R = 0 & \Phi^{**}_\varrho = -\Phi^*_\varrho \end{array}$$

waarbij de (rand)waarden die Φ, Φ^* of Φ^{**} in de punten van de cirkels R en ϱ aannemen, door de indices R en ϱ aangeduid zijn. In dat geval geldt inderdaad

$$\triangle' \Phi = \triangle' (\Phi^* + \Phi^{**}) = \triangle' \Phi^* + \triangle' \Phi^{**} = -2\,G\omega$$

$$\Phi_R = \Phi_R^* + \Phi_R^{**} = 0, \quad \Phi_\varrho = \Phi_\varrho^* + \Phi_\varrho^{**} = 0$$

zoodat $\Phi = \Phi^* + \Phi^{**}$ de gezochte oplossing van (2) is.

a. Ter bepaling van Φ^* wordt de oorsprong van het Cartesisch coördinatenstelsel gelegd in het middelpunt van den cirkel R en de y-as door de middelpunten van de cirkels ϱ (zie fig. 2).

Een oplossing van de vergelijking

$$\triangle' \Phi^* = -2 G\omega \quad \dots \dots \dots \quad (4)$$

is:

$$\Phi^* = -G\omega/2\,(y^2 + z^2) + C. \quad \dots \dots \quad (5)$$

De constante C is bepaald door den eisch dat voor $y^2 + z^2 = R^2$, $\Phi^* = \Phi \ = 0$ zij; men vindt:

$$\Phi^* = G\omega/2\,(R^2 - y^2 - z^2). \quad \dots \dots \dots \quad (6)$$

De waarden van Φ^* in de punten van de cirkels ϱ kan men als functie van y of z vinden, al naar men z of y elimineert uit (6) en de vergelijkingen van de cirkels ϱ. Overigens zullen wij dit niet uitvoeren, daar wij de vergelijking (6) nog zullen transformeeren op andere coördinaten.

b. Het vinden van een oplossing van de vergelijking

$$\triangle' \Phi^{**} = 0 \qquad \left(\triangle' \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \quad \dots \dots \quad (7)$$

die aan de randvoorwaarden $\Phi_R^{**} = 0$ en $\Phi_\varrho^{**} = -\Phi_\varrho^*$ voldoet, gaat het gemakkelijkst, wanneer wij gebruik maken van bi-polaire coördinaten (zie fig. 2).

$$\text{De transformatie } u = R \tanh v/2 \quad \dots \dots \quad (8)$$

waarin $\qquad u = y + iz \qquad$ en $\qquad v = \eta + i\zeta,$

geeft de transformatieformules:

$$y = R\frac{\sinh \eta}{\cosh \eta + \cos \zeta} \qquad z = R\frac{\sin \zeta}{\cosh \eta + \cos \zeta} \quad \dots \quad (9)$$

en doet (7) overgaan in:

$$\triangle' \Phi^{**} = 0 \qquad \left(\triangle' \equiv \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right) \dots \dots \quad (7')$$

Stelt men η of ζ constant, en elimineert ζ resp. η uit de vergelijkingen (9), dan vindt men de vergelijkingen van de, in fig. 2 geteekende, cirkelbundels in het y—z-vlak. Een strook in het η—ζ-vlak, begrensd door twee waarden van ζ die 2π verschillen, wordt op het geheele y—z-vlak afgebeeld. Door deze strook zóó te kiezen dat $-\pi \leqq \zeta \leqq +\pi$, zorgen wij ervoor dat, voor elke waarde van η in het gebied van de doorsnede, ζ een continue functie is van z. Zooals uit het voorgaande volgt wordt de dwarsdoorsnede

van de staaf in het η—ζ-vlak afgebeeld op een rechthoekig gebied, zoodat de randvoorwaarden van het probleem zich in de nieuwe coördinaten eenvoudiger laten formuleeren.

Ter bepaling van de randwaarden $\Phi^{**}_{\varrho} = -\Phi^{*}_{\varrho}$, schrijven wij eerst Φ^{*} als functie van η en ζ:

$$\Phi^{*} = G\,\omega\,R^{2}\,\frac{\cos\zeta}{\cosh\eta + \cos\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

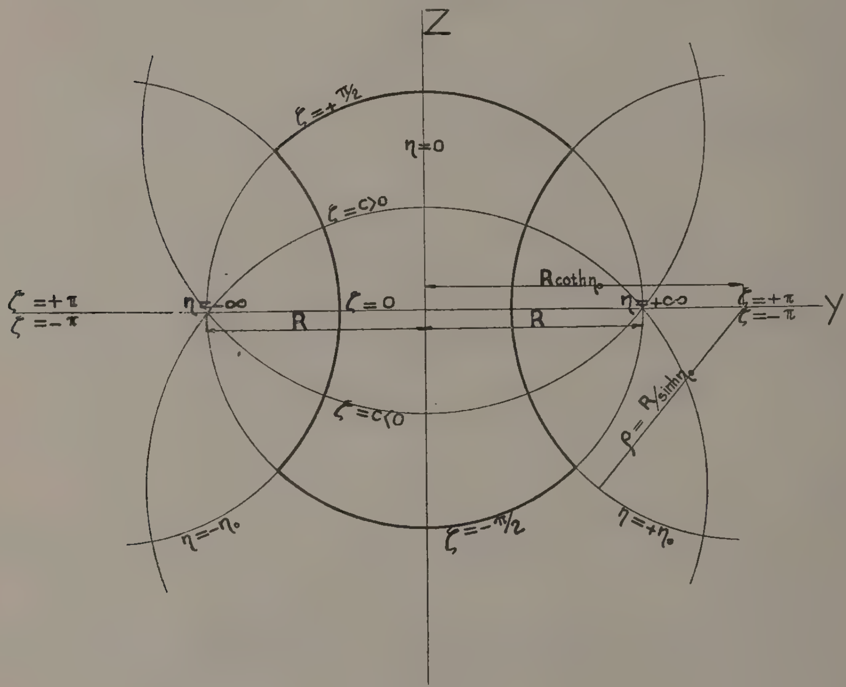

Fig. 2.

De cirkels ϱ worden bepaald door de twee waarden $\eta = \pm\,\eta_0$, zoodat

$$\Phi^{*}_{\varrho} = G\,\omega\,R^{2}\,\frac{\cos\zeta}{\cosh\eta_0 + \cos\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

en dus

$$\Phi^{**}_{\varrho} = -\,G\,\omega\,R^{2}\,\frac{\cos\zeta}{\cosh\eta_0 + \cos\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Stellen wij nu als oplossing van vergelijking (7′):

$$\Phi^{**} = \sum_{n=1}^{\infty}{}' \; a_n \cosh n\eta \cos n\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

waarvan elke term afzonderlijk aan (7′) voldoet (en waarin het accent bij het som-teeken aangeeft dat alleen oneven n-waarden in aanmerking

worden gebracht), dan ziet men terstond dat aan de randvoorwaarde $\Phi^{**}_R = 0$ automatisch voldaan is, zooals blijkt wanneer $\zeta = \pm \pi/2$ — overeenkomend met $y^2 + z^2 = R^2$ — gesteld wordt. Om de oplossing Φ^{**} aan te passen aan de voorwaarde (12) voeren wij een hulpfunctie Ψ in, die in het interval $-\pi/2 \leqq \zeta \leqq +\pi/2$ met Φ^{**}_ϱ overeenkomt en in de gebieden $-\pi \leqq \zeta \leqq -\pi/2$ resp. $+\pi/2 \leqq \zeta \leqq +\pi$ gedefinieerd is door $\Psi(\zeta) = -\Psi(\zeta+\pi)$ resp. $\Psi(\zeta) = \Psi(\zeta - \pi)$.

Een dergelijke functie kan ontwikkeld worden in een FOURIER-reeks van de gedaante $\Psi = \sum\limits_{n=1}^{\infty}{}' \; a_n \cos n\zeta$, waarin het accent wederom aangeeft dat alleen over oneven waarden van n gesommeerd wordt.

Zooals bekend zijn de coëfficiënten a_n bepaald door

$$\left.\begin{aligned} a_n = 1/\pi \int\limits_{-\pi}^{+\pi} \Psi \cos n\zeta \, d\zeta = -4/\pi \, G\omega R^2 \int\limits_0^{\pi/2} \frac{\cos\zeta \cos n\zeta}{\cosh\eta_0 + \cos\zeta} d\zeta = \\ = -4/\pi \, G\omega R^2 I_n \end{aligned}\right\} \quad . \quad . \quad (14)$$

met

$$I_n = \int\limits_0^{\pi/2} \frac{\cos\zeta \cos n\zeta}{\cosh\eta_0 + \cos\zeta} d\zeta.$$

zoodat

$$\Psi = -4/\pi \, G\omega R^2 \sum_{n=1}^{\infty}{}' \; I_n \cos n\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Wordt nu over het interval $-\pi/2 \leqq \zeta \leqq +\pi/2$ Φ^{**}_ϱ (13) geïdentificeerd met Ψ (15) door te stellen:

$$\Phi^{**}_\varrho = \sum_{n=1}^{\infty}{}' \alpha_n \cosh n\eta_0 \cos n\zeta = -4/\pi \, G\omega R^2 \sum_{n=1}^{\infty}{}' I_n \cos n\zeta$$

dan vindt men voor den coëfficiënt α_n

$$\alpha_n = -4/\pi \, G\omega R^2 \frac{I_n}{\cosh n\eta_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

zoodat

$$\Phi^{**} = -4/\pi \, G\omega R^2 \sum_{n=1}^{\infty}{}' \frac{I_n}{\cosh n\eta_0} \cosh n\eta \cos n\zeta. \quad . \quad . \quad (17)$$

De gezochte functie $\Phi = \Phi^* + \Phi^{**}$ kan men dus, als besluit van deze overwegingen, als volgt neerschrijven:

$$\Phi = G\omega R^2 \left(\frac{\cos\zeta}{\cosh\eta + \cos\zeta} - 4/\pi \sum_{n=1}^{\infty}{}' \frac{I_n}{\cosh n\eta_0} \cosh n\eta \cos n\zeta \right). \quad (18)$$

De grootste in de dwarsdoorsnede optredende schuifspanning wordt langs de cirkels ϱ aangetroffen. Haar richting is die van de raaklijn aan den

omtrek, haar grootte vindt men met behulp van de eerste afgeleide van Φ in de richting van de normaal op het contour:

$$\tau_\varrho = \left(\frac{d\Phi}{dn}\right)_{\eta=\eta_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Wegens de symmetrische gedaante van de doorsnede is het geoorloofd in het volgende uitsluitend positieve waarden van η en ζ te beschouwen. Voor elke eindige waarde van η_0 is dan volgens (19) τ_ϱ een functie van ζ, waarvan het maximum bepaald moet worden. Het zal blijken dat dit maximum niet altijd optreedt in het punt $\eta = \eta_0$, $\zeta = 0$, waar men het zou kunnen verwachten.

Voor (19) schrijven wij:

$$\tau_\varrho = \left(\frac{d\Phi}{dn}\right)_{\eta=\eta_0} = \left(\frac{\partial\Phi}{\partial\eta}\frac{d\eta}{dn}\right)_{\eta=\eta_0} + \left(\frac{\partial\Phi}{\partial\zeta}\frac{d\zeta}{dn}\right)_{\eta=\eta_0}.$$

en merken op dat

$$\left(\frac{d\zeta}{dn}\right)_{\eta=\eta_0} = 0 \text{ en } \left(\frac{d\eta}{dn}\right)_{\eta=\eta_0} = \frac{1}{\left(\frac{du}{dv}\right)_{\eta=\eta_0}} = 1/R\,(\cosh\eta_0 + \cos\zeta) \quad (20)$$

daarbij gebruik makend van de transformatieformule (8).

Met (18) berekent men nu verder gemakkelijk:

$$\left(\frac{\partial\Phi}{\partial\eta}\right)_{\eta=\eta_0} = G\,\omega\,R^2\,\frac{\cos\zeta\,\sinh\eta_0}{(\cosh\eta_0 + \cos\zeta)^2} + 4/\pi \sum_{n=1}^{\infty}{}' \, n\,I_n \tanh n\,\eta_0 \cos n\,\zeta) \quad (21)$$

zoodat

$$\left.\begin{aligned} \varkappa' = \tau_\varrho / G\,\omega\,R = \frac{\cos\zeta\,\sinh\eta_0}{\cosh\eta_0 + \cos\zeta} + 4/\pi\,(\cosh\eta_0 + {} \\ + \cos\zeta) \sum_{n=1}^{\infty}{}' \, n\,I_n \tanh n\,\eta_0 \cos n\,\zeta \end{aligned}\right\} \quad . \quad . \quad (22)$$

In nr. 3 zullen wij het maximum van $\varkappa' = f(\zeta)$ aangeven met $\varkappa$:

$$\varkappa = \tau_{\varrho\,(\max)} / G\,\omega\,R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Ook de stijfheid van de as kan, zooals wederom bekend verondersteld zij, bepaald worden met behulp van de kennis van Φ uit de formule:

$$W = 2 \int\!\!\int \Phi\,dy\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

waarin W het door de wringspanningen opgeleverde wringend moment voorstelt en de dubbelintegraal zich over de geheele doorsnede uitstrekt. Daar $\Phi = \Phi^* + \Phi^{**}$, geldt:

$$W = 2 \int\!\!\int \Phi^*\,dy\,dz + 2 \int\!\!\int \Phi^{**}\,dy\,dz. \quad . \quad . \quad . \quad . \quad (25)$$

De berekening van den eersten term van (25) levert geen bezwaren en verloopt — met inachtneming van (6) — het gemakkelijkst bij gebruik van poolcoördinaten (zie fig. 3)

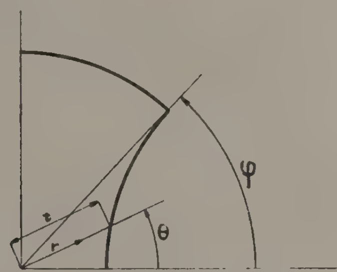

Fig. 3.

$$2\iint \Phi^* \, dy \, dz = 2\iint \Phi^* r \, dr \, d\theta =$$

$$= 4\,G\omega \left\{ \int_{\varphi}^{\pi/2} d\theta \int_0^R r\,(R^2 - r^2)\,dr + \int_0^{\varphi} d\theta \int_0^t r\,(R^2 - r^2)\,dr \right.$$

waarin $t = (R/\cos\varphi)(\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi})$.

Na uitwerking van de elementaire integralen volgt hieruit:

$$2\iint \Phi^* \, dy \, dz = G\omega R^4 \{(\pi/2 - \varphi)(1 + 3\tan^4\varphi) + \tan\varphi - 3\tan^3\varphi\}. \quad (26)$$ [3]

De tweede term van (25) kan met behulp van de integraalstelling van Green

$$\iint (U \triangle V - V \triangle U)\, dy\, dz = \oint \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) ds \quad . \quad (27)$$

getransformeerd worden, waarbij wij ons houden aan de gebruikelijke afspraken omtrent de positieve richtingen langs normaal en kromme. Stellen wij:

$$U = \Phi \text{ en } V = \Phi^{**}$$

dan is

$$\iint (\Phi \triangle \Phi^{**} - \Phi^{**} \triangle \Phi)\, dy\, dz = \oint \left(\Phi \frac{\partial \Phi^{**}}{\partial n} - \Phi^{**} \frac{\partial \Phi}{\partial n} \right) ds$$

of, daar $\triangle \Phi^{**} = 0$ en $\triangle \Phi = -2\,G\omega$:

$$2\,G\omega \iint \Phi^{**}\, dy\, dz = \oint \Phi \frac{\partial \Phi^{**}}{\partial n}\, ds - \oint \Phi^{**} \frac{\partial \Phi}{\partial n}\, ds.$$

---

[3]) In de analoge formule in het werk van Gronwall is een kleine onnauwkeurigheid ingeslopen, zonder zijn resultaten te beïnvloeden.

De eerste integraal uit het rechter lid valt weg, daar $\Phi = 0$ in de punten van den omtrek; de tweede integraal uit dat lid levert langs den cirkel R geen bijdrage, zoodat:

$$2\,G\omega \int\!\!\int \Phi^{**}\,dy\,dz = -4 \int_0^{\pi/2} \Phi^{**} \frac{\partial \Phi}{\partial \eta}\,d\zeta. \quad . \quad . \quad . \quad . \quad (28)$$

Men vindt

$$\left. \begin{aligned} 2 \int\!\!\int \Phi^{**}\,dy\,dz = -4\,G\omega R^4 \Bigg\{ \int_0^{\pi/2} \frac{\sinh \eta_0 \cos^2 \zeta}{\cosh \eta_0 + \cos \zeta)^3}\,d\zeta + \\ + 4/\pi \int_0^{\pi/2} \sum_{n=1}^{\infty}{}' \, n\,I_n \tanh \eta_0 \frac{\cos \zeta \cos n\zeta}{\cosh \eta_0 + \cos \zeta}\,d\zeta \Bigg\} \end{aligned} \right\} \quad (29)$$

waaruit, na verwisseling van sommatie en integratie en uitwerking van de elementaire integraal, volgt:

$$\left. \begin{aligned} 2 \int\!\!\int \Phi^{**}\,dy\,dz = G\,\omega R^4 \Big\{ -(\pi/2-\varphi)(6\tan^4\varphi + 2\tan^2\varphi) + 6\tan^3\varphi - \\ -16/\pi \sum_{n=1}^{\infty}{}' \, n\,I_n^2 \tanh n\,\eta_0 \Big\} \end{aligned} \right\}. \quad (30)$$

Hierbij is, teneinde aansluiting met (26) te verkrijgen, gebruik gemaakt van de betrekkingen $\sinh \eta_0 = 1/\tan \varphi$ en $\cosh \eta_0 = 1/\sin \varphi$, die men aan de hand van fig. 2 gemakkelijk verifieert.

Men vindt dus tenslotte:

$$\left. \begin{aligned} \mu = W/G\,\omega R^4 = \{(\pi/2-\varphi)\,(1-2\tan^2\varphi - 3\tan^4\varphi) + \\ + \tan\varphi + 3\tan^3\varphi - 16/\pi \sum_{n=1}^{\infty}{}' \, n\,I_n^2 \tanh n\,\eta_0\} \end{aligned} \right\} \quad . \quad . \quad (31)$$

Van de grootheden $\varkappa$ (23) en μ (31) levert de laatste een directe maat voor de stijfheid van de as, de eerste is slechts in zóóverre van beteekenis, dat zij ons in staat stelt een derden factor $\lambda = \mu/\varkappa$ te bepalen, die de grootste schuifspanning als functie van het wringend moment levert.

Voor de berekening van de normaalspanningen in een axiaal getrokken of gedrukte as, die wij slechts volledigheidshalve hier bespreken, dient men het oppervlak van de doorsnede te kennen. Maakt men weer gebruik van poolcoördinaten, dan vindt men:

$$F = \int\!\!\int r\,dr\,d\theta = R^2\,\{(\pi - 2\varphi)\,(1 - \tan^2\varphi) + 2\tan\varphi\} \quad . \quad . \quad (32)$$

De bij buigbelasting belangrijke grootheden zijn de traagheidsmomenten en weerstandsmomenten om de y- en z-as.

$$I_y = \int\int r^3 \sin^2\theta \, dr \, d\theta \qquad W_y = I_y/R$$

$$I_z = \int\int r^3 \cos^2\theta \, dr \, d\theta \qquad W_z = I_z/R \cos\varphi$$

Werkt men de integralen uit, dan vindt men:

$$\left.\begin{aligned} I_y &= R^4 \{(\pi/4-\varphi/2)(1-\tan^4\varphi) - 1/6\tan\varphi + 1/2\tan^3\varphi + 2/3\sin\varphi\cos\varphi\} \\ I_z &= R^4 \{(\pi/4-\varphi/2)(1-4\tan^2\varphi - 5\tan^4\varphi) + \\ &\qquad + 7/6\tan\varphi + 5/2\tan^3\varphi - 2/3\sin\varphi\cos\varphi\} \end{aligned}\right\} . \quad (33)$$

Tenslotte is de minimum traagheidsstraal, die een rol speelt bij knikberekeningen, te bepalen uit:

$$i_{min} = i_z = \sqrt{I_z/F}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

3. *Berekening.*

De resultaten van de berekeningen van alle in aanmerking komende grootheden ($\varkappa$, λ, μ, F, I_y, I_z, W_y, W_z, i_{min}) zijn voor de φ-waarden 0°, 10° ... 90° in tabel- en grafiekvorm ondergebracht in nr. **4.** Behalve de goniometrische functies van φ en ξ, en de hyperbolische functies van η_0, is daarbij gebruik gemaakt van de kennis van de integralen I_n. De goniometrische en hyperbolische functies kunnen in tabellen gevonden worden, de integralen I_n echter moeten, voor de met de gekozen φ-overeenkomende η_0-waarden, telkens voor een serie n-waarden berekend worden. Zijn de integralen I_n in voldoende aantal bekend, dan kunnen we voor elk punt der cirkels ϱ den factor $\varkappa'$ en dus ook zijn maximum $\varkappa$ berekenen. Volledigheidshalve is in de tweede kolom van tabel 1 (zie nr. **4**) de waarde van $\varkappa'$ in het punt $\eta = \eta_0$, $\zeta = 0$ opgenomen. Door vergelijking met de eerste kolom blijkt dan direct dat de waarde in dit punt niet steeds de maximale waarde vertegenwoordigt.

Bij de berekening van I_n maken wij voor groote waarden van η_0 ($\cosh\eta_0 > 2$) gebruik van de volgende ontwikkeling:

$$I_n = \sum_{p=1}^{\infty} (-1)^{p+1} \, 1/\cosh^p \eta_0 \int_0^{\pi/2} \cos^p\zeta \cos n\zeta \, d\zeta. \quad . \; . \; . \; . \quad (35)$$

Voor kleinere waarden van $\cosh\eta_0$ convergeert deze reeks te langzaam om met voordeel voor de berekening gebruikt te kunnen worden. Om voor deze waarden I_n te berekenen, maken wij gebruik van het volgende:

$$I_{n-2} + I_{n+2} = \int_0^{\pi/2} \frac{\cos\zeta\cos(n-2)\zeta}{\cosh\eta_0 + \cos\zeta} d\zeta + \int_0^{\pi/2} \frac{\cos\zeta\cos(n+2)\zeta}{\cosh\eta_0 + \cos\zeta} d\zeta =$$

$$= \int_0^{\pi/2} \frac{\cos\zeta\cos n\zeta}{\cosh\eta_0 + \cos\zeta} d\zeta + \int_0^{\pi/2} \frac{\cos 3\zeta\cos n\zeta}{\cosh\eta_0 + \cos\zeta} d\zeta.$$

In de tweede integraal van het derde lid voeren we in:

$$\cos 3\zeta = 4\cos^3\zeta - 3\cos\zeta$$

en vinden dan

$$I_{n-2} + I_{n+2} = 4\int_0^{\pi/2} \frac{\cos^3\zeta \cos n\zeta}{\cosh\eta_0 + \cos\zeta}\, d\zeta - 2 I_n.$$

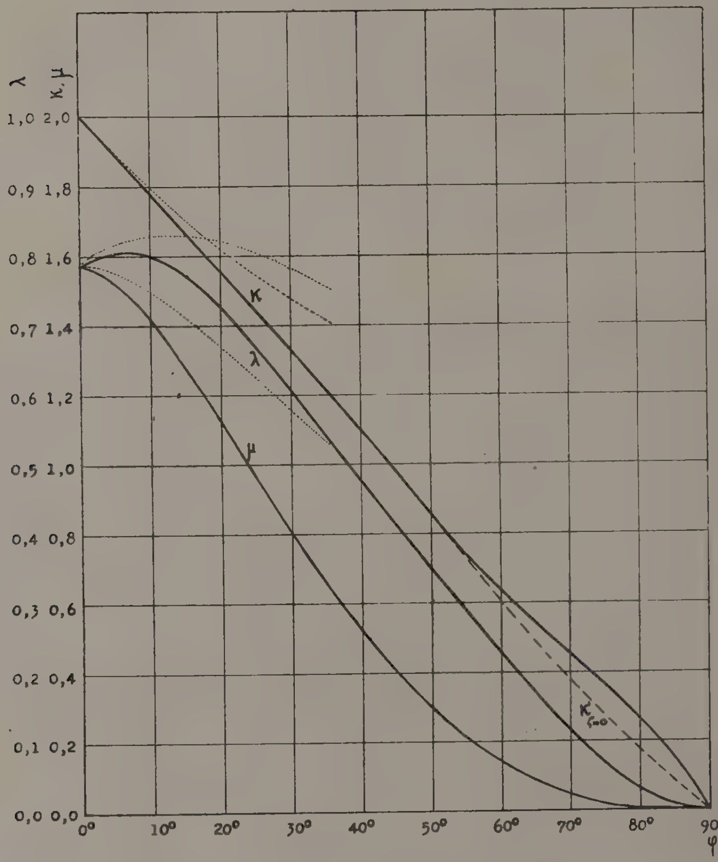

Fig. 4. $\varkappa$, λ en μ als functie van φ.

Door uitdeelen van de breuk $\dfrac{\cos^3\zeta \cos n\zeta}{\cosh\eta_0 + \cos\zeta}$ en uitwerken van de (elementaire) integralen komt er:

$$\int_0^{\pi/2} \frac{\cos^3\zeta \cos n\zeta}{\cosh\eta_0 + \cos\zeta}\, d\zeta = \cosh^2\eta_0\, I_n - \cosh\eta_0 \int_0^{\pi/2} \cos\zeta \cos n\zeta\, d\zeta +$$

$$+ (-1)^{\frac{n+1}{2}} \frac{2}{n\,(n^2-4)}$$

zoodat

$$\left. \begin{aligned} I_{n-2} + I_{n+2} - (4 \cosh^2 \eta_0 - 2)\, I_n = (-1)^{\frac{n+1}{2}} \frac{8}{n\,(n^2-4)} - \\ -4 \cosh \eta_0 \int_0^{\pi/2} \cos \zeta \cos n\zeta \, d\zeta \end{aligned} \right\} \quad . \quad . \quad (36)$$

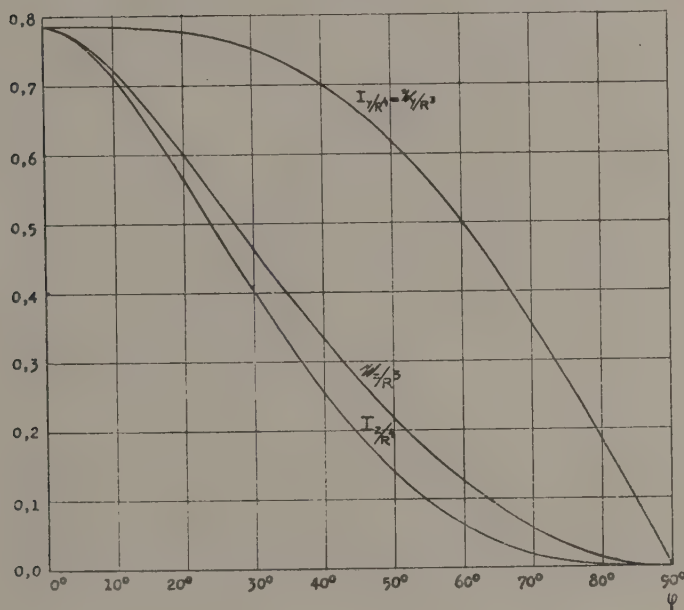

Fig. 5. I_y, W_y, I_z en W_z als functie van φ.

waarbij de integraal in het tweede lid — zooals bekend — voor $n \neq 1$ nul is en voor $n = 1$ gelijk aan $\pi/4$.

Met behulp van deze vergelijking kan men, als

$$I_1 = 1 - \frac{\pi}{2 \sin \varphi} + (\pi/2 - \varphi)(\tan \varphi + 1/\tan \varphi). \quad . \quad . \quad . \quad (37)$$

eenmaal berekend is, alle andere integralen I_3, I_5 . . . . . bepalen. Immers voor $n = 1$ geeft (36):

$$I_{-1} - (4 \cosh^2 \eta_0 - 2)\, I_1 + I_3 = 8/3 - \pi \cosh \eta_0$$

of

$$I_3 = (4 \cosh^2 \eta_0 - 3)\, I_1 + 8/3 - \pi \cosh \eta_0, \text{ daar } I_{-1} = I_1.$$

Door (36) daarna voor $n = 3$ op te schrijven vindt men I_5 enz.

Daar I_1 uit den aard der zaak slechts in een beperkt aantal decimalen wordt becijferd, sluipt bij de berekening van I_3, I_5 . . . . . op den duur een

niet toelaatbare onnauwkeurigheid in, die een gevolg is van het feit dat de eerste term in het tweede lid van (36) en de derde term in het eerste lid van dezelfde vergelijking een zeer klein verschil vertoonen, dat door uiterst kleine wijzigingen van I_n sterk beïnvloed wordt. Om aan dit bezwaar

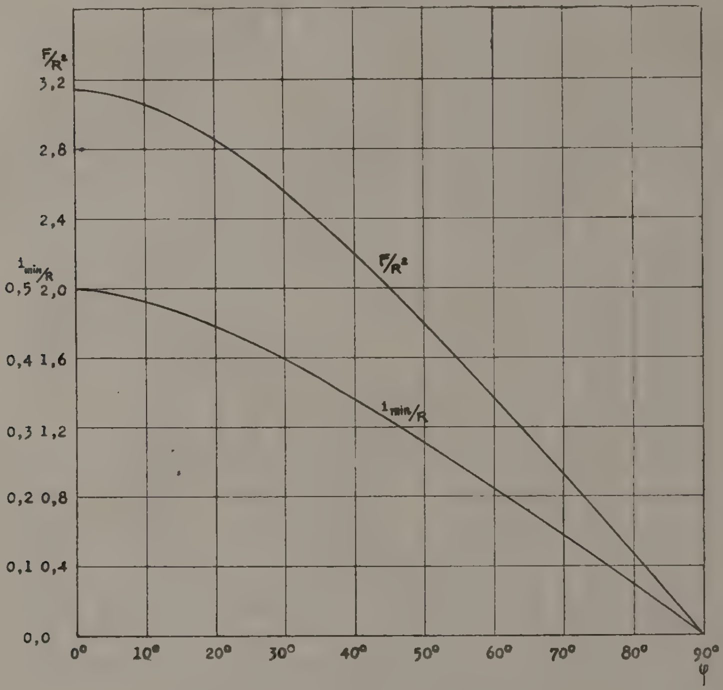

Fig. 6. F en i_{min} als functie van φ.

tegemoet te komen, schrijft men in de gevallen waar dit optreedt $(30° < \varphi < 60°)$ (36) in den vorm

$$I_n = \frac{(-1)^{\frac{n+1}{2}} \dfrac{8}{n\,(n^2-4)} - I_{n-2} - I_{n+2}}{-(4\cosh^2\eta_0 - 2)} \quad (n \neq 1) \ldots \quad (38)'$$

Nadat dan, hetzij volgens (35), hetzij volgens (36) een serie benaderingswaarden $I_3, I_5 \ldots$ bepaald is, wordt (38) als een iteratieformule gebruikt, waarin in het tweede lid de reeds gevonden waarden worden gesubstitueerd en het eerste lid de nauwkeuriger waarden I_n $(n = 3, 5 \ldots)$ levert. Met de nieuw verkregen waarden kan zoo noodig het iteratieproces herhaald worden. In tabel 1 zijn voor een aantal waarden van φ de integralen I_n tot $n = 27$ vereenigd.

Voor de berekening van $\varkappa'$ (22) spelen de reeksen

$$\sum_{n=1}^{\infty}{}' \, n\, I_n \tanh n\eta_0 \cos n\zeta \quad \ldots \ldots \ldots \quad (39)$$

een belangrijke rol. Hiervoor schrijven wij:

$$\sum_{n=1}^{\infty}{}' \, n I_n \tanh n\eta_0 \cos n\zeta = \sum_{n=1}^{\infty}{}' \, n I_n \cos n\zeta - \sum_{n=1}^{\infty}{}' \, n I_n \cos n\zeta (1 - \tanh n\eta_0). \quad (40)$$

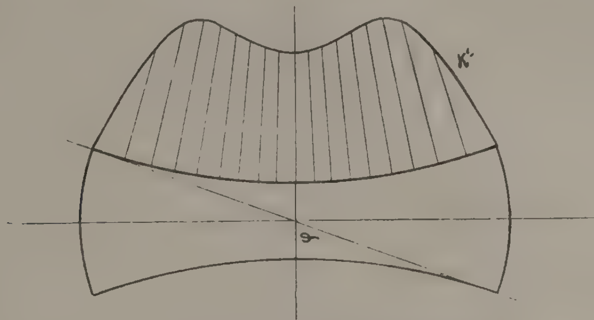

Fig. 7. $\varkappa'$ voor $\varphi = 70°$, als functie van ζ.

De eerste reeks in het tweede lid van (40) is te berekenen met behulp van een uit (36) af te leiden formule. Vermenigvuldigt men (36) met $n \cos n\zeta$ en sommeert over oneven waarden van n van 1 tot ∞, dan vindt men na korte berekening

$$\sum_{n=1}^{\infty}{}' \, n I_n \cos n\zeta = \frac{(2I_1 + \cosh \eta_0) \cos \zeta - \sum_{n=1}^{\infty}{}' (-1)^{\frac{n+1}{2}} \frac{8}{n^2-4} \cos n\zeta}{4(\cosh^2 \eta_0 - \cos^2 \zeta)}$$
$$- \sin 2\zeta \left\{ \frac{(2I_1 + \cosh \eta_0) \sin \zeta - \sum_{n=1}^{\infty}{}' (-1)^{\frac{n+1}{2}} \frac{8}{n(n^2-4)} \sin n\zeta}{4(\cosh^2 \eta_0 - \cos^2 \zeta)^2} \right\}.$$

Wanneer men dan nog opmerkt, dat

$$\left.\begin{aligned} \sum_{n=1}^{\infty}{}' (-1)^{\frac{n+1}{2}} \frac{8}{n(n^2-4)} \sin n\zeta &= 2 \sin \zeta + (1 + \cos 2\zeta) \ln \frac{1+\sin\zeta}{\cos\zeta} \\ \sum_{n=1}^{\infty}{}' (-1)^{\frac{n+2}{2}} \frac{8}{(n^2-4)} \cos n\zeta &= 4 \cos \zeta - 2 \sin 2\zeta \ln \frac{1+\sin\zeta}{\cos\zeta} \end{aligned}\right\} \quad 0 \leqq \zeta \leqq \frac{\pi}{2}$$

is men in staat om de waarde van $\sum_{n=1}^{\infty}{}' \, nI_n \cos n\zeta$ met iederen gewenschten graad van nauwkeurigheid te bepalen.

De tweede reeks in het tweede lid van (40) convergeert snel en kan eenvoudig door sommatie van de eerste 5 of 6 termen gevonden worden. Kent men eenmaal het verloop van $\varkappa'$ langs de cirkels η, dan vindt men het maximum $\varkappa$ van $\varkappa'$, in die gevallen waarin de grootste schuifspanning niet in het punt $\eta = \eta_0$, $\zeta = 0$ optreedt, door in de omgeving van het maximum $\varkappa' = f(\zeta)$ door een geschikt gekozen algebraïsche functie te benaderen. Ter illustratie is het verloop van $\varkappa'$ voor $\varphi = 70°$ geteekend (zie 4).

De sommatie van de reeksen $\sum_{n=1}^{\infty}{}' \, nI_n^2 \tanh n\eta_0$, die vereischt wordt bij

TABEL 1.

$$I_n = \int_0^{\pi/2} \frac{\cos x \ \cos nx}{(1/\sin \varphi + \cos x)}\, dx.$$

$n \backslash \varphi$	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
1	+0.00000	+0.11895	+0.20864	+0.27681	+0.32852	+0.36727	+0.39540	+0.41449	+0.42557	+0.42920
3	−0.00000	−0.00317	−0.01002	−0.01803	−0.02585	−0.03274	−0.03833	−0.04241	−0.04489	−0.04572
5	+0.00000	+0.00056	+0.00204	+0.00405	+0.00625	+0.00835	+0.01016	+0.01154	+0.01240	+0.01269
7	−0.00000	−0.00019	−0.00071	−0.00147	−0.00235	−0.00322	−0.00399	−0.00459	−0.00497	−0.00510
9	+0.00000	+0.00009	+0.00033	+0.00070	+0.00112	+0.00155	+0.00194	+0.00225	+0.00245	+0.00252
11	−0.00000	−0.00005	−0.00018	−0.00038	−0.00061	−0.00086	−0.00108	−0.00126	−0.00138	−0.00142
13	+0.00000	+0.00003	+0.00011	+0.00023	+0.00037	+0.00052	+0.00066	+0.00077	+0.00085	+0.00087
15	−0.00000	−0.00002	−0.00007	−0.00015	−0.00024	−0.00034	−0.00044	−0.00054	−0.00056	−0.00057
17	+0.00000	+0.00001	+0.00005	+0.00010	+0.00017	+0.00024	+0.00030	+0.00035	+0.00038	+0.00040
19	−0.00000	−0.00001	−0.00003	−0.00007	−0.00012	−0.00017	−0.00022	−0.00025	−0.00028	−0.00029
21	+0.00000	+0.00001	+0.00003	+0.00005	+0.00009	+0.00013	+0.00016	+0.00019	+0.00021	+0.00021
23	−0.00000	−0.00001	−0.00002	−0.00004	−0.00007	−0.00010	−0.00012	−0.00015	−0.00016	−0.00016
25	+0.00000	+0.00000	+0.00002	+0.00003	+0.00005	+0.00008	+0.00010	+0.00011	+0.00013	+0.00013
27	−0.00000	−0.00000	−0.00001	−0.00003	−0.00004	−0.00006	−0.00008	−0 00009	−0.00010	−0.00010

de berekening van μ, geeft voor de beschouwde η_0-waarden geen aanleiding tot moeilijkheden.

4. *Resultaten.*

De resultaten onzer berekeningen zijn in de fig. 4, 5, 6, 7 en in tabel 2 weergegeven. Hieraan kan nog het volgende worden ontleend:

$\varkappa$ wordt voor $\varphi < 70^\circ$ met groote benadering voorgesteld door

$$\varkappa = 2 - \frac{4\varphi}{\pi} \quad \ldots\ldots\ldots\ldots \quad (41)$$

De grootste fout is ongeveer 4,5 %.

TABEL 2.

φ	$\varkappa$	$\varkappa'_{\eta=\eta_0,\ \zeta=0}$	λ	μ	F/R^2	I_y/R^4 W_y/R^3	I_z/R^4	W_z/R^3	i_{min}/R
0°	2.0000	2.0000	0.7854	1.5708	3.1416	0.7854	0.7854	0.7854	0.5000
10°	1.7818	1.7818	0.7981	1.4220	3.0584	0.7848	0.7133	0.7243	0.4829
20°	1.5644	1.5644	0.7238	1.1323	2.8477	0.7779	0.5645	0.6007	0.4452
30°	1.3388	1.3388	0.6059	0.8112	2.5510	0.7541	0.4006	0.4626	0.3963
40°	1.1020	1.1020	0.4763	0.5249	2.1947	0.7038	0.2536	0.3311	0.3399
50°	0.8556	0.8556	0.3506	0.3000	1.7967	0.6209	0.1390	0.2163	0.2781
60°	0.6321	0.6079	0.2261	0.1429	1.3697	0.5037	0.0617	0.1233	0.2122
70°	0 4475	0.3767	0.1097	0.0491	9.9232	0.3555	0.0189	0.0554	0.1432
80°	0.2635	0.1777	0.0283	0.0075	0.4647	0.1840	0.0024	0.0139	0.0722
90°	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Vergelijken wij $\varkappa$, λ en μ voor kleine φ-waarden met de, in fig. 4 gestippelde, overeenkomstige grootheden voor een as met één orthogonaal-cirkelvormige insnijding, dan kunnen wij de volgende opmerkingen maken:

1. De sterkte is van beide assen ongeveer dezelfde ($\varkappa$).
2. De stijfheid van de eene as neemt — bij eenzelfde toename van φ — ongeveer tweemaal zoo sterk af als die van de andere (μ).

Tenslotte toont het verloop van λ in fig. 4 dat men spiesleuven bij voorkeur met een $\varphi \approx 8^\circ$ zal uitvoeren, om de sterkte van het materiaal zooveel mogelijk te benutten.

Delft, 1—5—'45.

Mechanics. — *De stabiliteit van de tolbewegingen van* STAUDE. By O. BOTTEMA. (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of October 27, 1945.)

In 1894 ontdekte STAUDE [1]) dat een willekeurig, in een van het zwaartepunt G verschillend punt O ondersteund en in het zwaarteveld geplaatst vast lichaam om elk van ∞^1 assen door O een rotatie met constante hoeksnelheid kan uitvoeren. Een dergelijke as moet daarbij verticaal worden geplaatst. Deze *assen van* STAUDE vormen een quadratische kegel en VAN DER WOUDE [2]) heeft veel later aangetoond dat de totale verzameling dezer assen, bij veranderlijke plaats van O, niets anders is dan het quadratisch complex van de hoofdtraagheidsassen van het lichaam.

De *stabiliteit* van de rotaties van STAUDE is onderzocht door GRAMMEL [3]), die met behulp van de methode der kleine trillingen er in slaagde de stabiliteitsvoorwaarden in één eenvoudige gedaante te brengen en in zekere zin het vraagstuk volledig oploste. Hij zag zich daarbij echter genoodzaakt het probleem anders te stellen dan in de meest voor de hand liggende vorm. De vraag, die zich voordoet, luidt blijkbaar in beginsel zo: door de traagheidsmomenten α, β en γ ten opzichte van de hoofdassen door O en verder door de coördinaten ξ, η en ζ van het zwaartepunt G ten opzichte van deze assen, is het stelsel dynamisch volledig gekarakteriseerd en is in het bijzonder de quadratische kegel der assen van STAUDE bepaald; welke dezer assen zijn stabiel? Het blijkt nu, dat men bij de beantwoording van deze vraag dusdanige algebraïsche complicaties ontmoet, dat het vrij hopeloos schijnt een overzichtelijke oplossing te bereiken [4]).

GRAMMEL heeft daarom de vraag anders gesteld. Hij neemt het punt O en de traagheidsmomenten α, β, γ als gegeven aan en laat de coördinaten ξ, η, ζ van G voorlopig onbepaald. Daarna kiest hij een willekeurige as l door O en gaat na, waar men het punt G zou moeten kiezen, opdat l een as van STAUDE van het systeem wordt. Voor G wordt zo een bepaalde meetkundige plaats F gevonden. Onderzocht wordt nu verder in welke deel-

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[1]) STAUDE, Über permanente Rotationsachsen bei der Bewegung eines schweren Körpers um einen festen Punkt, Journ. f. reine u. angew. Math. **113**, 318—334 (1894).

[2]) VAN DER WOUDE, Über die Staudeschen Kreiselbewegungen, Math. Zeitschr. **16**, 170—172 (1923).

[3]) GRAMMEL, Die Stabilität der Staudeschen Kreiselbewegungen, Math. Zeitschr. **6**, 124—142 (1920).

[4]) GRAMMEL, t.a.p. pg. 129. Vgl. ook METTLER, Periodische und asymptotische Bewegungen des unsymmetrischen schweren Kreisels, Math. Zeitschr. **43**, 59—100 (1937), in het bijzonder pg. 81.

verzameling F' van F het punt G moet liggen, opdat l een stabiele as is. Het aldus gewijzigde stabiliteitsprobleem wordt volledig opgelost.

In het volgende keren wij tot de oorspronkelijke probleemstelling terug, maar beperken ons daarbij tot een zeer bijzonder geval, n.l. tot dat, waarbij G op een der hoofdassen door O ligt. De kegel van STAUDE ontaardt dan in de beide hoofdvlakken door deze as en het stabiliteitsprobleem heeft een betrekkelijk eenvoudige oplossing, vooral dank zij het feit, dat een der drie door GRAMMEL opgestelde condities steeds vervuld blijkt te zijn. Ligt G niet op een hoofdas, maar wel in een hoofdvlak van O, dan is de kegel der rotatie-assen eveneens ontaard en het lijkt niet uitgesloten ook in dit algemener geval resultaten te verkrijgen. Wij gaan daar hier echter niet nader op in.

Als inleiding geven wij een kort overzicht van de methode van GRAMMEL. Kiest men het aan het lichaam verbonden stelsel van hoofdassen door O als coördinatenstelsel, zijn x_0, y_0, z_0 de coördinaten van een op de afstand 1 verticaal boven O gelegen vast punt der ruimte en is ω de constante hoeksnelheid om de verticaal geplaatste as van STAUDE, dan ontstaan door specialisatie van de betrekkingen van EULER de volgende bewegingsvergelijkingen:

$$\left.\begin{aligned}(\beta-\gamma)\,\omega^2 y_0 z_0 &= \eta z_0-\zeta y_0\\(\gamma-\alpha)\,\omega^2 z_0 x_0 &= \zeta x_0-\xi z_0\\(\alpha-\beta)\,\omega^2 x_0 y_0 &= \xi y_0-\eta x_0\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

De componenten der hoeksnelheid zijn $p_0 = \omega x_0$, $q_0 = \omega y_0$ en $r_0 = \omega z_0$ en men kan de grootheden x_0, y_0, z_0, p_0, q_0, r_0 als coördinaten der beweging beschouwen. GRAMMEL geeft aan deze grootheden kleine veranderingen, zodat zij de waarden $\bar{x} = x_0 + x$, $\bar{p} = p_0 + p$ enz. verkrijgen en noemt de betrokken beweging stabiel als x, y, z, p, q, r kleine grootheden blijven. Door substitutie in de algemene bewegingsvergelijkingen van een vast lichaam en na toepassing van het in de theorie der kleine trillingen gebruikelijke procédé, ontstaat een stelsel van zes lineaire differentiaalvergelijkingen. Stelt men de oplossingen evenredig met $e^{\varrho t}$, dan ontstaat een frequentievergelijking, die van de derde graad in ϱ^2 blijkt te zijn en bovendien de wortel $\varrho^2 = 0$ blijkt te bezitten. Er blijft een vierkantsvergelijking in ϱ^2 over, welke, zal de rotatie stabiel zijn, twee verschillende reële, negatieve wortels moet hebben. Deze vergelijking ziet er als volgt uit:

$$g_0\varrho^4 + g_1\varrho^2 + g_2 = 0 \quad \ldots\ldots\ldots \quad (2)$$

waarbij

$$\left.\begin{aligned}g_0 &= \alpha\beta\gamma\\g_1 &= \Sigma(\beta\gamma u + \alpha l^2\\g_2 &= \Sigma[\alpha(vw-\lambda^2) + ul^2 + 2\lambda mn]\end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

De sommatie moet daarbij over drie, door cyclische verwisseling ontstane

termen worden uitgestrekt. De in (3) voorkomende grootheden hangen daarbij op de volgende wijze van de gegevens af:

$$\left.\begin{aligned} u &= (\beta-\gamma)(y_0^2-z_0^2)\,\omega^2-\eta\,y_0-\zeta\,z_0 \\ l &= (\beta+\gamma-\alpha)\,\omega\,x_0 \\ \lambda &= [\zeta-(\gamma-\alpha)\,\omega^2 z_0]\,y_0=[\eta+(\alpha-\beta)\,\omega^2 y_0]\,z_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

terwijl v, w, m, n, μ, ν door cyclische verwisseling worden gevonden. De stabiliteitscondities luiden dus:

$$g_1>0, \quad g_2>0, \quad k\equiv g_1^2-4\,g_0 g_2>0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Wij merken nog op, dat de vergelijking van de kegel van STAUDE, zoals door eliminatie van ω^2 uit (1) blijkt, als volgt luidt:

$$(\beta-\gamma)\,y_0 z_0\,\xi+(\gamma-\alpha)\,z_0 x_0\,\eta+(\alpha-\beta)\,x_0 y_0\,\zeta=0 \quad . \quad . \quad . \quad (6)$$

Een as van STAUDE is daarbij gekarakteriseerd door de verhoudingen $x_0 : y_0 : z_0$. Bij elke as behoort, zoals uit (1) blijkt, in het algemeen één bepaalde waarde van ω^2. Tevens is voor elke as bepaald welke der beide halfstralen door O bij de permanente rotatie verticaal naar boven gericht moet zijn, en wel door de voorwaarde dat ω^2 een positieve waarde moet hebben.

Wij beschouwen het bijzondere geval, waarbij $\eta=\zeta=0$, dus dat, waarbij het zwaartepunt G op een hoofdtraagheidsas OX door O ligt. Zoals uit (6) blijkt, is dan de kegel van STAUDE ontaard in de beide hoofdvlakken $y_0=0$ en $z_0=0$. Tot de assen van STAUDE behoort ook de snijlijn $y_0=z_0=0$, dus de as OG; kiest men deze als wentelingsas, dan is de bijbehorende hoeksnelheid onbepaald en de vraag, hoe de stabiliteit van de hoeksnelheid afhangt, is door GRAMMEL volledig opgelost. Wij kunnen dus deze as buiten beschouwing laten. Verder sluiten wij de assen $y_0=x_0=0$ en $z_0=x_0=0$ uit (aangezien zij met de hoeksnelheid $\omega=\infty$ overeenkomen) en veronderstellen voorts, dat α, β, γ onderling verschillend zijn. Wij richten onze aandacht op de overige assen van STAUDE; zoals uit (1) blijkt, behoort bij elk van hen één welbepaalde hoeksnelheid ω. Uit (1) volgt immers voor $z_0=0$ dat $x_0=\dfrac{\xi}{(\alpha-\beta)\,\omega^2}$ is, terwijl y_0 willekeurig blijft, terwijl voor $y_0=0$ de waarde van z_0 willekeurig is en $x_0=\dfrac{\xi}{(\alpha-\gamma)\,\omega^2}$. Veronderstellen wij $\xi>0$, wat zonder de algemeenheid te schaden, gebeuren kan, dan heeft men dus, daar $\omega^2>0$ is, in het eerste geval $x_0\,(\alpha-\beta)>0$, in het tweede $x_0\,(\alpha-\gamma)>0$. Duiden wij die halfstraal op een as van STAUDE, welke bij de permanente rotatie verticaal naar boven moet worden geplaatst, aan als de positieve halfstraal, dan vullen deze positieve halfstralen dus zowel in $y_0=0$ als in $z_0=0$ een halfvlak. Deze halfvlakken zijn blijkbaar $y_0=0$, $x_0>0$ en $z_0=0$, $x_0>0$ als $\alpha>\beta$ en $\alpha>\gamma$; voor $\alpha<\beta$, $\alpha>\gamma$ zijn de halfvlakken $y_0=0$, $x_0>0$

en $z_0 = 0$, $x_0 < 0$ en voor $\alpha < \beta$, $\alpha < \gamma$ vindt men $y_0 = 0$, $x_0 < 0$ en $z_0 = 0$, $x_0 < 0$.

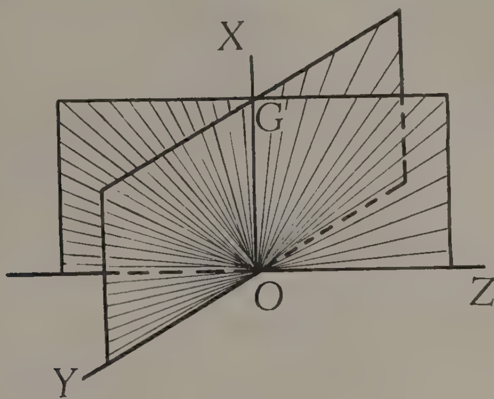


Fig. 1*a*.

In fig. 1 is de ligging der positieve halfstralen geschetst, respectievelijk voor het geval dat het zwaartepunt G ligt op de hoofdtraagheidsas, die bij het grootste, het middelste en het kleinste traagheidsmoment behoort. In het

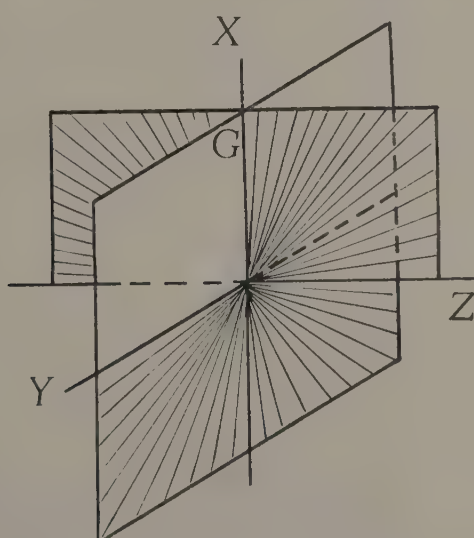


Fig. 1*b*.

eerste geval ligt dus bij alle permanente rotaties het zwaartepunt boven het steunpunt, in het derde geval ligt het er steeds beneden, terwijl in het middelste geval beide situaties voorkomen.

Wij willen thans nagaan om welke assen stabiele rotatie plaats heeft.

Voor de grootheden (4) vindt men, als $z_0 = 0$ is en $y_0 = \frac{y_1}{\omega^2}$ wordt gesteld:

$$\left.\begin{array}{lll} u = \dfrac{(\beta-\gamma)\, y_1^2}{\omega^2} & v = \dfrac{(\beta-\gamma)\, \xi^2}{(\alpha-\beta)^2\, \omega^2} & w = -\dfrac{(\alpha-\beta)\, y_1^2}{\omega^2} \\ l = \dfrac{(\beta+\gamma-\alpha)\, \xi}{(\alpha-\beta)\, \omega} & m = \dfrac{(\gamma+\alpha-\beta)\, y_1}{\omega} & n = 0 \\ \lambda = 0 & \mu = 0 & \nu = -\dfrac{\beta-\gamma\, \xi\, y_1}{(\alpha-\beta)\, \omega^2} \end{array}\right\} \quad (7)$$

zodat men voor de coëfficiënten (3) heeft:

$$g_1 = \frac{\alpha\, \xi^2\, A}{(\alpha-\beta)^2\, \omega^2} + \frac{\beta\, y_1^2\, B}{\omega^2} \quad (8)$$

$$g_2 = \frac{y_1^2\, (\beta-\gamma)}{(\alpha-\beta)^2\, \omega^4} \left\{ \xi^2\, C - y_1^2\, \beta\, (\alpha-\beta)^3 \right\} \quad (9)$$

waaruit volgt

$$k = g_1^2 - 4 g_0\, g_2 = \frac{\alpha^2\, \xi^4\, A^2}{(\alpha-\beta)^4\, \omega^4} + \frac{2\, \alpha\, \beta\, \xi^2 y_1^2\, D}{(\alpha-\beta)^2\, \omega^4} + \frac{\beta^4\, y_1^4\, \varepsilon^2}{\omega^4} \quad (10)$$

Hierbij is

$$A = \alpha^2 + \beta^2 - 2\, \alpha\, \beta - 2\, \alpha\, \gamma + 3\, \beta\, \gamma \quad (11)$$

$$B = \beta^2 + 2\, \alpha\, \gamma - \alpha\, \beta - \beta\, \gamma \quad (12)$$

$$C = \alpha^2 + 2\, \beta^2 + 2\, \gamma^2 - 3\, \alpha\, \beta \quad (13)$$

$$\left.\begin{array}{l} D = A\, B - 2\, \gamma\, (\beta-\gamma)\, C = -\alpha^3 \beta + 2\, \alpha^3 \gamma + 3\, \alpha^2 \beta^2 - 5\, \alpha^2 \beta\, \gamma - \\ -2\, \alpha^2 \gamma^2 - 3\, \alpha\, \beta^3 + 5\, \alpha\, \beta^2 \gamma + 2\, \alpha\, \beta\, \gamma^2 + \beta^4 - 2\, \beta^3 \gamma + \beta^2 \gamma^2 - 4\, \beta\, \gamma^3 + 4\, \gamma^4 \end{array}\right\} \quad (14)$$

$$E = \alpha - \beta + \gamma. \quad (15)$$

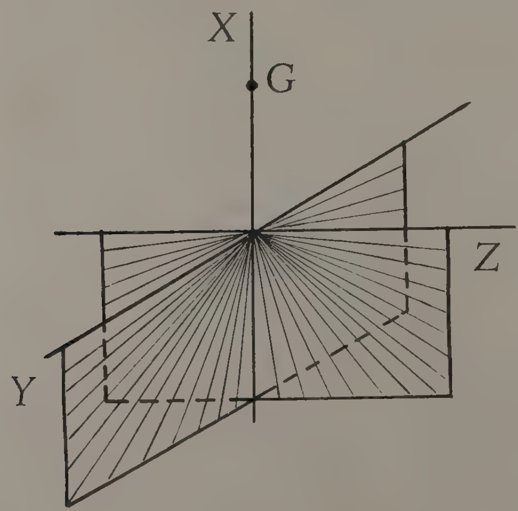


Fig. 1c.

Elke as van STAUDE uit het vlak $z_0 = 0$ wordt aangegeven door een bepaalde waarde van y_1, want men heeft na eliminatie van ω^2 de betrekking $\frac{y_0}{x_0} = \frac{(\alpha-\beta)\,y_1}{\xi}$. Voor de stabiliteit van een as is nodig en voldoende, dat g_1, g_2 en k positief zijn, en wij hebben na te gaan voor welke waarden van y_1 zulks het geval is. Natuurlijk is het antwoord afhankelijk van de onderlinge verhoudingen der gegeven traagheidsmomenten α, β en γ. Daar deze aan de ongelijkheden $\alpha < \beta + \gamma$ etc. voldoen, voeren wij de grootheden u_i $(i = 1, 2, 3)$ in door

$$2u_1 = -\alpha + \beta + \gamma, \qquad 2u_2 = \alpha - \beta + \gamma, \qquad 2u_3 = \alpha + \beta - \gamma,$$

waarvoor dus geldt $u_i > 0$, terwijl

$$\alpha = u_2 + u_3, \qquad \beta = u_3 + u_1, \qquad \gamma = u_1 + u_2 \quad . \quad . \quad . \quad (16)$$

Men vindt dan

$$A = 2u_1^2 - u_2^2 + u_2 u_3 + u_3 u_1 - u_1 u_2.$$

Duidt men u_i als driehoekscoördinaten, dan stelt $A = 0$ een kegelsnede voor, die door de middens van U_1U_2 en U_2U_3 gaat, en in het laatste aan de rechte $u_2 = u_3$ raakt. Zij is in fig. 2 geschetst voor zover zij binnen de coördinatendriehoek ligt. Wij komen tot de conclusie dat A zowel positief als negatief kan zijn. Voor punten buiten de kegelsnede is $A > 0$.

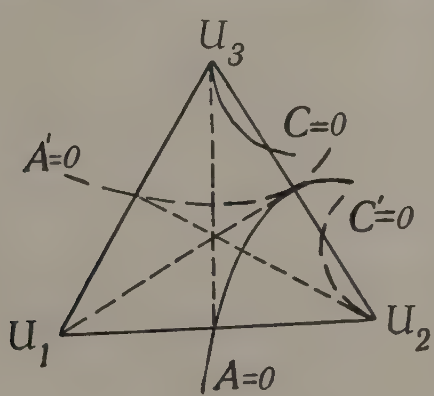


Fig. 2.

De uitdrukking B gaat door (16) over in $2(u_1u_3 + u_2^2)$, waaruit volgt dat B voor alle in aanmerking komende waarden van α, β en γ positief is.

Voor C vindt men $4u_1^2 + 3u_2^2 - u_2u_3 + u_3u_1 + u_1u_2$, zodat $C = 0$ een kegelsnede voorstelt, die in U_3 aan $u_1 = u_2$ raakt en van dit punt tot $(0, 1, 3)$ binnen de driehoek verloopt, zoals in de figuur is geschetst.

De uitdrukking D blijkt in factoren te kunnen worden ontbonden. Men heeft nl.

$$D = (\alpha - \beta + \gamma)\, D_1$$

waarbij

$$D_1 = -\alpha^2\beta + 2\alpha^2\gamma + 2\alpha\beta^2 - 2\alpha\beta\gamma - 4\alpha\gamma^2 - \beta^3 + \beta^2\gamma + 4\gamma^3.$$

Wij onderzoeken nu eerst de functie k. Men heeft, als

$$v_1 = \frac{\alpha\,\xi^2}{(\alpha-\beta)^2\,\omega^2}, \qquad v_2 = \frac{\beta y_1^2(\alpha-\beta+\gamma)}{\omega^2}$$

wordt gesteld,

$$k = A^2 v_1^2 + 2D_1 v_1 v_2 + \beta^2 v_2^2$$

zodat

$$k = (A v_1 - \beta v_2)^2 + 2(D_1 + \beta A) v_1 v_2.$$

Nu is

$$D_1 + \beta A = 2\gamma(\alpha^2 - 2\alpha\beta - 2\alpha\gamma + 2\beta^2 + 2\gamma^2) = 2\gamma\{(-\alpha+\beta+\gamma)^2 + (\beta-\gamma)^2\}.$$

Deze uitdrukking is dus altijd positief. Daar $v_1 > 0$ en $v_2 > 0$ en in het vlak $y_0 = 0$ overeenkomstige uitkomsten gelden, heeft men: *als het zwaartepunt G op een hoofdtraagheidsas door O ligt, is de derde stabiliteitsvoorwaarde $k > 0$ voor elke as van* STAUDE *vervuld.*

De voorwaarde $g_1 > 0$ is blijkbaar steeds vervuld als $A > 0$. Als $A < 0$ dan is $g_1 > 0$ als

$$\frac{y_0^2}{x_0^2} > -\frac{\alpha A}{\beta B}.$$

Voor de bepaling van het teken van g_2 zijn van belang de uitdrukkingen $\beta-\gamma$, $\alpha-\beta$ en C. De voorwaarde $\beta-\gamma \gtrless 0$ is gelijkwaardig met $u_3 \gtrless u_2$. Uit fig. 2 blijkt dat uit $\beta-\gamma < 0$ volgt $C > 0$, terwijl voor $\beta-\gamma > 0$ zowel $C > 0$ als $C < 0$ kan gelden.

$\alpha-\beta \gtrless 0$ is gelijkwaardig met $u_2 \gtrless u_1$, zodat uit $\alpha-\beta < 0$ volgt $C > 0$. g_2 heeft een constant teken als $(\alpha-\beta)\ C < 0$; dit teken is dat van $(\beta-\gamma)\ C$. Voor $(\alpha-\beta)\ C > 0$ wisselt g_2 van teken en wel heeft g_2 voor

$$\frac{y_0^2}{x_0^2} < \frac{C}{\beta(\alpha-\beta)}$$

het teken van $C(\beta-\gamma)$ en voor de overige waarden het tegengestelde teken.

Wij zullen nu de verschillende mogelijkheden afzonderlijk beschouwen. Zij eerst $\alpha > \beta > \gamma$, d.w.z. $u_3 > u_2 > u_1$. Uit fig. 2 blijkt, dat dan $A > 0$ terwijl C zowel positief als negatief kan zijn. De voorwaarde $g_1 > 0$ is dus vervuld. g_2 heeft een constant teken, als $C < 0$; g_2 is dan negatief. Voor $C > 0$ wisselt g_2 van teken en wel is $g_2 > 0$ als

$$\frac{y_0^2}{x_0^2} < \frac{C}{\beta(\alpha-\beta)}.$$

Wij vinden dus: *als $\alpha > \beta > \gamma$ dan is voor $C < 0$ geen der rotaties stabiel; voor $C > 0$ zijn ze alleen stabiel als*

$$\frac{y_0^2}{x_0^2} < \frac{C}{\beta(\alpha-\beta)}.$$

Is $\alpha > \gamma > \beta$, dus $u_2 > u_3 > u_1$, dan is $C > 0$. Verder is $g_1 > 0$ als $A > 0$, terwijl voor $A < 0$ g_1 alleen positief is als

$$\frac{y_0^2}{x_0^2} < -\frac{\alpha A}{\beta B}.$$

Daar $(\alpha - \beta)\ C > 0$ is, heeft g_2 een wisselend teken, waarbij zij positief is voor

$$\frac{y_0^2}{x_0^2} > \frac{C}{\beta(\alpha-\beta)}.$$

Om na te gaan welke der getallen

$$-\frac{\alpha A}{\beta B} \text{ en } \frac{C}{\beta(\alpha-\beta)}$$

het grootst is, berekenen wij het verschil $BC + \alpha(\alpha - \beta)A$. Hiervoor vindt men

$$5\,u_2^4 - 2\,u_2^3\,u_3 + 3\,u_2^2\,u_1^2 - 2\,u_2\,u_1^3 + 2\,u_2^3\,u_1 - 2\,u_2\,u_1\,u_3^2 + \\ + 4\,u_1^2\,u_2\,u_3 + u_2^2\,u_3^2 + 6\,u_1^3\,u_3 + u_1^2\,u_3^2 + 8\,u_1\,u_2^2\,u_3 + 8\,u_1^2\,u_2^2.$$

De drie termen met negatieve coëfficiënt zijn elk kleiner dan de voorafgaande term, zodat de uitdrukking voor de in aanmerking komende waarden van u_i positief is. Daaruit volgt

$$-\frac{\alpha A}{\beta B} < \frac{C}{\beta(\alpha-\beta)},$$

zodat *wij hebben voor $\alpha > \gamma > \beta$: die assen zijn stabiel, waarvoor*

$$\frac{y_0^2}{x_0^2} > \frac{C}{\beta(\alpha-\beta)}.$$

Voor $\beta > \alpha > \gamma$, dus $u_3 > u_1 > u_2$, is $A > 0$ en $C > 0$, dus $g_1 > 0$. Daar $(\alpha - \beta)\ C < 0$ heeft g_2 een vast teken en wel is $g_2 > 0$. *In dit geval zijn dus alle assen stabiel.*

Voor $\beta > \gamma > \alpha$, dus $u_1 > u_3 > u_2$, is $A > 0$ en $C > 0$; dus is $g_1 > 0$ en daar $(\alpha - \beta)\ C < 0$ heeft g_2 constant teken en wel is $g_2 > 0$. *Ook nu zijn dus alle assen stabiel.*

Voor $\gamma > \alpha > \beta$ is $u_2 > u_1 > u_3$, zodat $C > 0$, terwijl A beide tekens kan hebben. Voor $A > 0$ is $g_1 > 0$, voor $A < 0$ zal g_1 nog positief zijn als

$$\frac{y_0^2}{x_0^2} > \frac{-\alpha A}{\beta B}.$$

Wat g_2 betreft, deze wisselt van teken en is positief voor

$$\frac{y_0^2}{x_0^2} > \frac{C}{\beta(\alpha-\beta)}.$$

Als boven vinden wij thans stabiliteit voor

$$\frac{y_0^2}{x_0^2} > \frac{C}{\beta(\alpha-\beta)}.$$

Als tenslotte $\gamma > \beta > \alpha$, dus $u_1 > u_2 > u_3$, dan is $A > 0$ en $C > 0$, $(\alpha - \beta)\ C < 0$, $(\beta - \gamma)\ C < 0$, zodat g_2 negatief is. *Geen as is thans stabiel.*

Het bovenstaande had steeds betrekking op de assen van STAUDE, gelegen in het vlak $z_0 = 0$. Voor het vlak $y_0 = 0$ gelden analoge resultaten, die men verkrijgen kan door β en γ met elkaar te verwisselen. Daarbij worden dus van belang de uitdrukkingen

$$A' = \alpha^2 + \gamma^2 - 2\alpha\gamma - 2\alpha\beta + 3\beta\gamma.$$
$$C' = \alpha^2 + 2\beta^2 + 2\gamma^2 - 3\alpha\gamma.$$

De krommen $A' = 0$ en $C' = 0$ zijn in fig. 2 eveneens geschetst. Vatten wij de resultaten thans samen, dan krijgen wij het volgende overzicht, waarin telkens de *stabiele* assen van STAUDE zijn aangegeven.

Geval		$y_0 = 0$	$z_0 = 0$
Ia	$\alpha > \beta > \gamma$	$\frac{z_0^2}{x_0^2} > \frac{C'}{\gamma(\alpha-\gamma)}$	$C < 0$ — $C > 0$ $\frac{y_0^2}{x_0^2} < \frac{C}{\beta(\alpha-\beta)}$
IIa	$\alpha > \gamma > \beta$	$C' < 0$ — $C' > 0$ $\frac{z_0^2}{x_0^2} < \frac{C'}{\gamma(\alpha-\gamma)}$	$\frac{y_0^2}{x_0^2} > \frac{C}{\beta(\alpha-\beta)}$
I*b*	$\beta > \alpha > \gamma$	$\frac{z_0^2}{x_0^2} > \frac{C'}{\gamma(\alpha-\gamma)}$	alle
II*b*	$\gamma > \alpha > \beta$	alle	$\frac{y_0^2}{x_0^2} > \frac{C}{\beta(\alpha-\beta)}$
Ic	$\beta > \gamma > \alpha$	—	alle
IIc	$\gamma > \beta > \alpha$	alle	—

De gevallen I^a en II^a, resp. I^b en II^b, alsmede I^c en II^c komen daarbij op hetzelfde neer en ontstaan door verwisseling van β met γ of van de vlakken

$y_0 = 0$ en $z_0 = 0$. Zij corresponderen resp. met de figuren 1^a, 1^b en 1^c. Voor de toestand 1^a is het antwoord op de vraag naar de stabiliteit het meest gecompliceerd. In een der beide halfvlakken door de X-as komen steeds stabiele assen voor; zij liggen buiten een bepaalde hoek, die de X-as tot bissectrice heeft. In het andere der halfvlakken zijn òf alle assen labiel, òf wel komen er stabiele assen voor, die een hoek opvullen, waarvan de X-as de bissectrice is.

In 1^b zijn al die assen stabiel, welke in het halfvlak liggen, dat het zwaartepunt G niet bevat. Bij de rotatie om deze assen ligt G beneden het steunpunt O. Ook in het andere halfvlak liggen stabiele assen, de labiele vullen een hoek waar OX binnen ligt.

In 1^c zijn ten slotte de assen uit het ene halfvlak stabiel, die uit het andere labiel.

Indien twee der traagheidsmomenten α, β en γ gelijk zijn, ondergaan de resultaten enige wijziging. Is $\alpha = \gamma$, dan blijkt uit (1) dat alleen het vlak $z_0 = 0$ assen van STAUDE bevat; die uit $y_0 = 0$ vervallen omdat bij elk daarvan $\omega = \infty$ zou behoren. Voor de assen uit $z_0 = 0$ gelden de stabiliteitscriteria van ons schema, waarbij nog opgemerkt moet worden, dat C in deze omstandigheden steeds positief is. Voor $\alpha = \beta$ gelden dezelfde uitkomsten na verwisseling van y_0 en z_0. Voor $\beta = \gamma$ ten slotte gelden de resultaten van het schema met dien verstande, dat C en C' ook hierbij alleen positieve waarden kunnen hebben.

Botany. — *De invloed van colchicine op den groei van den celwand van wortelharen.* By CHR^e. J. GORTER. (Communicated by Prof. G. VAN ITERSON Jr.)

(Communicated at the meeting of October 27, 1945.)

a. De invloed op de structuur van den wand.
b. De invloed op de chemische samenstelling van den wand.

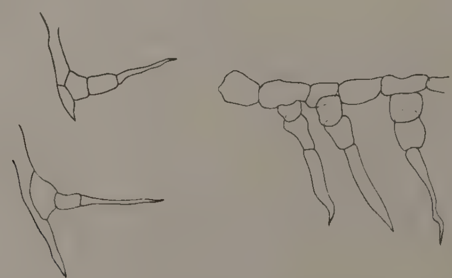
Methodiek: De proeven werden genomen met wortelharen van *Trianea bogotensis, Lepidium sativum, Avena sativa, Triticum vulgare, Brassica napus, Setaria italica, Commelina benghalensis* en *Hordeum vulgare.* Dit zijn planten, die wortelharen vormen, zoowel in vochtige lucht als in water. In sommige gevallen werd gewerkt met luchtwortelharen; deze hebben echter het bezwaar, dat ze, op het objectglas in water gebracht, barsten of aan den top eigenaardige verdikkingen vormen (ZACCHARIAS, 1891, SOKOLOWA, 1897). De luchtwortelharen werden gekweekt door de zaden te leggen op een poreuze plaat, doordrenkt met de oplossing. De colchicine-oplossingen werden gemaakt met leidingwater.

De vorm der met colchicine behandelde wortelharen.

Het doel van dit onderzoek was iets te weten te komen van de veranderingen, die de groei van den celwand door colchicine ondergaat.

Het is sedert eenige jaren bekend, dat groeiende plantendeelen onder invloed van colchicine een vormverandering ondergaan. Als voorbeeld geven wij foto 1 en figuur 1. De *Hyacinthus*-wortels op foto 1, die gegroeid zijn in colchicine-oplossing, vertoonen plaatselijk een alzijdige opzwelling. Deze is gelocaliseerd in de groeizône. De cellen van de haren op de okselknoppen van kiemplanten van *Helianthus annuus,* waarvan het hypocotyl is bestreken met colchicine-pasta, later een sterkeren groei in de breedte zien (fig. 1).

A.



B.

Fig. 1.

Haren op de jonge blaadjes der okselknoppen van kiemplanten van *Helianthus annuus.* A. van normaal materiaal. B. van met colchicine behandelde planten.

Ook bij wortelharen kan men door colchicine verschillende vormveranderingen zien optreden. Afbeeldingen hiervan geeft figuur 2. Blijkbaar heeft men hier te doen met groeistoornissen.

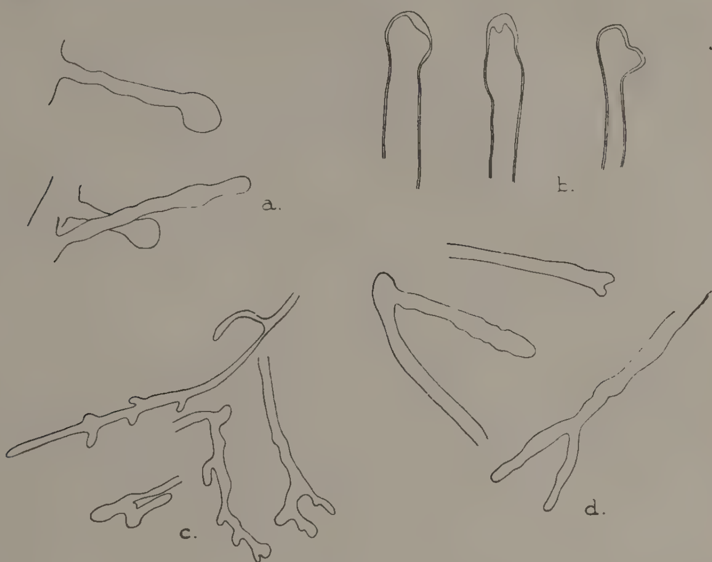


Fig. 2.
Vormveranderingen van wortelharen onder invloed van colchicine. a. *Lepidium sativum;* b. en d. *Avena sativa*, b. jong, d. oud; c. *Brassica napus.*

In het algemeen kunnen invloeden, die storend op den groei werken, bij wortelharen verschillende reacties teweegbrengen, die wij in drie categorieën willen samenvatten (zie de schematische figuur 3):

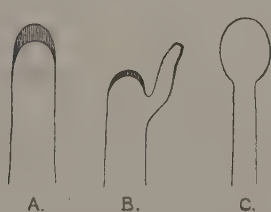


Fig. 3.
Vormveranderingen van toppen van wortelharen (schematisch). A. wandverdikking aan de top; B. wandverdikking aan de top, vorming van een zijtak; C. opzwelling van de top.

1e. (fig. 3A). Wandverdikkingen aan den top, welke verdikkingen veroorzaakt worden, doordat op die plaats door het protoplasma veel celwandmateriaal wordt afgezet, welk celwandmateraal niet wordt gebruikt voor den lengtegroei van de cel.

Dit celwandmateriaal bestaat niet uit callose, zooals wel eens gedacht

is. Hierop wijzen de volgende reacties: het kleurt niet met resoblauw, het lost niet op in geconcentreerd zoutzuur en ook niet in 1 % KOH-oplossing, het zwelt niet op met ammoniak, het kleurt daarentegen wel met congorood; het kleurt wel met rutheenrood, hetgeen wijst op de aanwezigheid van pectinestoffen; het lost niet op in Cuoxam en is dus geen zuivere cellulose. Waarschijnlijk hebben wij hier dus te doen met een combinatie van verschillende celwandstoffen. Een deel hiervan is cellulose, want de verdikking kleurt met chloorzinkjodium.

2e. (fig. 3B). Velerlei vormen van vertakkingen. Dit is een uitbreiding van het eerste geval. Aan den top kan de groei niet worden voortgezet, doch daaronder treedt nieuwe groei op, waardoor een zijtak ontstaat. Door een dergelijke groeiwijze, waarbij dus niet de top, maar een ander gedeelte van den celwand den groei voortzet, krijgen vele wortelharen vertakkingen of een slangvormig uiterlijk.

3e. (fig. 3C). Knopvormige opzwellingen aan het eind van het haar.

Vooral met de twee laatste verschijnselen, die ook als reactie op een colchicinebehandeling optreden, hebben wij ons beziggehouden.

Alvorens echter de structuur van den celwand van met colchicine behandelde wortelharen te beschrijven, moet iets gezegd worden van de structuur van den celwand van normale wortelharen.

De structuur van den celwand van normale wortelharen en hare mogelijke verklaring.

Om een indruk te krijgen van de ligging der micellen in den wand van een wortelhaar levert een onderzoek tusschen gekruiste nicols weinig op. De breking in licht, trillende in de richtingen evenwijdig aan de lengteas der cel en loodrecht daarop, is te weinig verschillend dan dat men dit verschil zichtbaar kan maken. Na kleuring met ClZnJ treedt er echter een sterk dichroïsme op. Op foto 2 ziet men een normaal wortelhaar van *Lepidium sativum*, gekleurd met dit reagens, in gepolariseerd licht met trillingsrichtingen, die loodrecht op elkaar staan. In het eerste geval is het haar donker gekleurd, in het andere licht gekleurd. Door vergelijking van dergelijke gekleurde wortelharen met vezels (b.v. rameh-vezels), waarvan bekend is, in welke richting de micellen in den wand (d. i. bij rameh een secundaire wandverdikking) liggen, kon worden vastgesteld, *dat in het cylindrische gedeelte van de celwanden van alle onderzochte normale wortelharen de meerderheid der micellen in de lengterichting, d.w.z. evenwijdig aan de lengteas der cel, zijn gerangschikt.*

Lengtegroei heeft bij wortelharen alleen plaats aan den uitersten top, zooals reeds door HABERLANDT (1887) en REINHARDT (1892) is vastgesteld. Aan wortelharen van *Trianea bogotensis*, gemerkt met fijn meniepoeder, kon ik deze waarneming bevestigen. Dit *groeiende* gedeelte van het wortelhaar zal een primairen wand moeten hebben. Het optische gedrag van den top was niet na te gaan. Het cylindrische deel van wortelharen

heeft een wand met een secundaire verdikking en het is waarschijnlijk, dat het optische gedrag daardoor wordt bepaald. De micellen zullen dan in dien secundairen wand in de lengterichting van de cel geplaatst zijn.

Door G. van Iterson Jr. (1936, 1942) is de theorie verkondigd, dat de micellen van celwandstof (die van gestrekten vorm zijn) bij de eerste afzetting van een celwandlaag gericht worden door de spanningen, die tengevolge van den turgordruk heerschen in het rustende deel van het protoplasma, waarin de micellen worden gevormd. Eenmaal afgezette micellen zouden zich door een soort kristallisatieproces kunnen vergrooten. Deze theorie verklaart o. a., waarom in de primaire laag van den zijwand eener cylindrische cel de micellen dwars op de lengterichting van de cel worden geplaatst; de spanningen in den wand van een cylinder. waarin een vloeistofdruk heerscht, zijn namelijk in de dwarsrichting gericht.

Het is verder een bekend feit, dat een celwand rekbaarder is in de richting loodrecht op de micellenrichting dan evenwijdig daarmee. Wanneer dus een primaire wandlaag, bijvoorbeeld door het toenemen van den turgordruk in de cel, wordt uitgerekt, zal dit het sterkste dwars op de micellen geschieden. Een primaire zijwand van een cylindrische cel zal dus het sterkst in de lengterichting der cel worden uitgerekt. De spanningen in de rustende laag van het protoplasma tegen zulk een primaire wandlaag zullen dan ook in de lengterichting verloopen. Door deze beschouwing laat zich verklaren, waarom in secundaire wandlagen van een cylindrische cel de micellen in de lengterichting geplaatst worden.

De structuur van den celwand van met colchicine behandelde wortelharen.

1. *Lepidium sativum.* Laat men wortelharen van deze plant zich ontwikkelen in een oplossing van colchicine of tegen een poreuze plaat. met deze oplossing doordrenkt, dan ontwikkelen de toppen zich abnormaal, zooals afgebeeld in fig. 3A en C.

Welke structuur vertoonen nu de wanden in deze abnormaal gevormde toppen?

Wij beschouwen eerst het geval, waarin een *wandverdikking* aan den top (als in fig. 3A) is gevormd, zooals is afgebeeld op foto 3A en B, waar het wortelhaar is gegroeid in 0.5 % colchicine-oplossing. De hierdoor veroorzaakte wandverdikking aan den top kleurt zich, evenals de rest van den celwand, met ClZnJ, paars. De photo's zijn genomen met gepolariseerd licht; voor photo A was de polarisatierichting van het licht 90° gedraaid ten opzichte van die voor photo B. In A ziet men een donkeren steel met een lichte topverdikking, in B een lichten steel met donkere topverdikking. Aangezien we nu weten, dat in den steel de meerderheid der micellen in de lengterichting gerangschikt is, blijkt hieruit dus. dat *in de topverdikking de meerderheid der micellen in de dwarsrichting ligt,* d.w.z. loodrecht op de lengteas van het wortelhaar.

Bezien wij nu in de tweede plaats het geval, waarin een *knotsvormige opzwelling aan den top* (als in fig. 3C) optreedt, d.i. de meest algemeene reactie van de wortelharen op een behandeling met colchicine (hetzij gegroeid in een oplossing of tegen een doordrenkte poreuze plaat). Op photo 4 ziet men een dergelijk wortelhaar, weer gekleurd met ClZnJ en daarna gefotografeerd met gepolariseerd licht. De steel is licht gekleurd, de opzwelling donker. Het blijkt dus, dat in de opzwelling de meerderheid der micellen in de dwarsrichting ligt tegengesteld aan de richting in den steel. Ook de foto's 5A en B laten hetzelfde verschijnsel zien: in A een donkere steel met lichtere opzwelling, in B een lichte steel met donkerder opzwelling.

In de opzwelling wordt dus onder invloed der colchicine de meerderheid der micellen in de dwarsrichting afgezet. De normale celwandgroei is daar gestoord.

2. *Avena sativa.* Op de colchicine reageeren deze wortelharen, zooals te zien is in figuur 2 b en d, n.l. door een onregelmatigen groei met dikwijls vertakkingen. In gepolariseerd licht na kleuring met ClZnJ ziet men in de wanden onregelmatige strepingen, zooals is waar te nemen op foto 6. Het maakt den indruk, alsof er aan den binnenkant tegen den celwand onregelmatige afzettingen van celwandmateriaal zijn gevormd, waardoor de celwanden gegolfd schijnen.

Deze streping is dikwijls schuin georiënteerd ten opzichte van de lengteas der cel en verloopt dan veelal in twee richtingen loodrecht op elkaar. Ook bij gewone belichting is deze streping bij gekleurde celwanden te zien, echter minder duidelijk. Bij de normale wortelharen treedt een dergelijke streping niet op, de wand is daar regelmatig paars gekleurd. De mogelijkheid werd overwogen, dat bij de colchicine-wortelharen de wand minder stevig zou zijn, zoodat bij opheffing van den turgor door de ClZnJ de wand zich zou plooien. Dit blijkt echter niet het geval te zijn, want, na plasmolyse van ongekleurde haren, ziet men geen vouwen in den wand, terwijl de plasmolytische verkorting bij de normale en de colchicinewortelharen dezelfde is. Wij blijken hier dus met een structuur van den celwand te doen te hebben, die men het beste aldus kan omschrijven:

Een behandeling met colchicine heeft ten gevolge, dat de micellen, die tegen den primairen wand worden afgezet en de secundaire verdikkingslagen vormen, niet in de lengterichting gerangschikt worden, maar onregelmatig komen te liggen en wel overheerschend in een richting schuin t. o. v. de lengteas der cel en een richting loodrecht daarop (zie fig. 4).

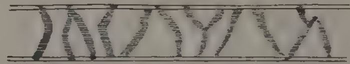


Fig. 4.
Wortelhaar van *Avena sativa* na behandeling van den wortel met colchicine. Streping van den wand (schematisch).

Ditzelfde verschijnsel heb ik aangetroffen bij *Triticum, Tradescantia* en *Brassica*, echter nooit bij *Lepidium* e. a.

3. *Trianea bogotensis*. Bij de normale wortelharen van deze plant ziet men na kleuring met ClZnJ slechts een zwak dichroïsme, er bestaat dus slechts een geringe voorkeur bij de meerderheid der micellen in de secundaire wandlagen voor de lengterichting. Bij deze haren ziet men dikwijls een netvormige structuur aan den binnenkant van den celwand. De secundaire verdikkingslagen zijn hier dus niet als regelmatige gladde lagen met in de lengterichting gerangschikte micellen afgezet.

Trianea-wortelharen reageeren op de colchicine (b.v. na een verblijf van 3 dagen in een oplossing van 0.1 % colchicine) vrijwel niet; een enkel wortelhaar vertoont in het midden een kleine opzwelling, terwijl de wand daar eenigszins dikker is, evenals aan den top. Er is dus slechts geringe groeistoornis. Zelfs na 16 dagen in deze oplossing was geen duidelijker reactie te zien. Alleen de zeer jonge wortelharen in het stadium, dat zij als een kleine bol uit de epidermiscel te voorschijn komen, zijn bij behandeling der worteltjes met colchicine breeder dan de normale en vertoonen soms een begin van vertakking, n.l. een dubbelen top.

Wij kunnen dus verschillende reacties onderscheiden van de wortelharen op colchicine, n.l.:

a. Een opzwelling van den top, waarbij de meerderheid der cellulosemicellen in de dwarsrichting komt te liggen (*Lepidium*).

b. Een onregelmatige groei van den celwand, waarbij de micellen in verschilende richtingen komen te liggen, dikwijls in een spiraalvorm (*Avena, Triticum, Tradescantia*).

c. Vrijwel geen reactie. Dit bij wortelharen, waarbij de meerderheid der micellen ongericht ligt (*Trianea*).

Met verwijzing naar hetgeen op pag. 3 en 4 gezegd is over de ligging der micellen in de normale wortelharen, zouden wij de volgende verklaring kunnen geven voor de meeste der hier beschreven verschijnselen:

Veronderstelt men, dat de werking der colchicine daarin bestaat, dat de primaire celwand meer plastisch dan elastisch wordt, dan zal het verschil in uitrekbaarheid in de richting der micellen en loodrecht daarop verdwijnen en dan zullen de micellen van de secundaire wandlagen zich in dezelfde richting als die van de primaire wandlaag kunnen afzetten. Ook kan men zich voorstellen, dat er een dusdanige verandering in de vastheid van de primaire wandlaag optreedt, dat de afzetting van de micellen in de secundaire wandlaag zonder voorkeursrichting plaats vindt.

De spiraalvormige afzettingen laten zich hiermee niet verklaren, hiervoor zal men wellicht de theorie, die G. van Iterson Jr. (1943) (in aansluiting aan C. Nageli) ter verklaring van den spiraalvormigen groei in het algemeen geopperd heeft, kunnen te hulp roepen.

De chemische samenstelling van met colchicine behandelde celwanden.

De mogelijkheid bestaat, dat colchicine inwerkt op één (of meer) der

celwandstoffen. Om een chemische analyse van celwanden mogelijk te maken, werd een groot aantal (400 stuks) bollen van *Hyacinthus orientalis* gezet op een oplossing van colchicine van 0,025 % en een even groot aantal ter contrôle op water. Van iedere partij werden ongeveer 5000 worteltoppen geoogst, resulteerende in een hoeveelheid droge stof van ± 6 gram van ieder soort. De colchicinewortels waren sterk opgezwollen (zie foto 1). Het drogestof-gehalte van de normale en de colchicinewortels is gelijk (6,7 % en 6,61 %). Het volume der colchicinewortels was grooter dan van de normale wortels en dit grootere volume blijkt dus geheel te wijten te zijn aan een grootere hoeveelheid in de wortels opgenomen water.

De worteltoppen werden gebracht in alcohol 96 %, daarna bij 30° C. gedroogd en vermalen in den Peppinkmolen.

Aan dit materiaal werd door den heer J. J. BREEN, assistent aan het Lab. v. Technische Botanie te Delft, analyses uitgevoerd, die de volgende uitkomsten opleverden:

	normale wortels	colchicinewortels
Cellulose	15,10 %	16,90 %
Lignine	4,93 „	8,97 „
Pectine (als tetragalacturonzuur)	12,51 „	20,84 „
Pentosanen	5,83 „	5,14 „
Asch	7,95 „	8,63 „

Als celwandstoffen werd dus in het eerste geval 46 %, in het tweede geval 60 % der droge stof teruggevonden. Een poging om de overige celbestanddeelen te bepalen, werd aan de colchicinewortels ondernomen. Hierbij werd gevonden:

In water onoplosbare stof	64,27 %
„ „ „ asch	1,82 %
Stikstofgehalte	6,17 %
„ „ , na uittrekken met water	6,40 %.

De in water oplosbare stof bevat 5,6 % stikstof = 35 % eiwit. Verder zijn 16,6 % monosen en 4,2 % maltose-achtige producten bepaald (door vergisting met *Torula monosa*, *Torula lactosa* en *Saccharomyces cerevisiae*). Als men de onderlinge verhouding, waarin de eigenlijke celwandstoffen voorkomen, berekent, krijgt men:

	normale wortels	colchicinewortels
Cellulose	39,4 % [1])	32,6 %
Lignine	12,8 „	17,3 „
Pectine	32,6 „	40,2 „
Pentosanen	15,2 „	9,9 „

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[1]) De asch is als celwandstof niet meegerekend.

Hieruit blijkt, dat bij de colchicinewortels het cellulosegehalte geringer is geworden, evenals dat der pentosanen; het lignine- en pectinegehalte is vooruitgegaan.

We mogen dus wel concludeeren, dat de werking der colchicine op de celwanden hieruit bestaat, dat er pectine gevormd wordt in plaats van cellulose.

Samenvatting.

Colchicine heeft op de groeiende celwanden invloed, zoowel op de structuur als op de chemische samenstelling.

Na kleuring met ClZnJ kon in de wanden van wortelharen een dichroïsme worden vastgesteld. Dit dichroïsme leert ons iets omtrent de ligging der micellen in deze wanden. Bij normale wortelharen ligt in de wanden van het cylindrische, niet meer in de lengte groeiende, deel van het wortelhaar het meerendeel der micellen in de lengterichting gerangschikt, d.w.z. evenwijdig aan de lengteas der cel.

Onder invloed van colchicine vertoonen wortelharen groeistoornissen. aanleiding gevende tot het voorkomen van topverdikkingen, knotsvormige opzwellingen aan den top, vertakkingen enz. Hierbij worden de micellen onregelmatig afgezet, n.l. òf in de dwarsrichting, òf in schuine richting, òf geheel ongericht.

Een chemische analyse van met colchicine behandelde wortels van hyacinthen liet zien, dat het gehalte aan cellulose en pentosanen lager is dan bij normale wortels, het pectine- en ligninegehalte hooger.

Op theoretische gronden wordt aangenomen, dat de werking der colchicine bestaat in een verandering in de mechanische eigenschappen der wanden, in dien zin, dat die wanden minder elastisch en meer plastisch worden.

Dit onderzoek werd verricht in het Laboratorium voor Technische Botanie der Technische Hoogeschool te Delft, onder leiding van Prof. Dr. G. van Iterson Jr., wien ik hierbij gaarne dank wil zeggen voor zijn hulp. Tevens spreek ik mijn erkentelijkheid uit jegens het Delftsch Hoogeschoolfonds, dat voor het verrichten van dit onderzoek finantieelen steun verleende.

Zusammenfassung.

Colchizin beeinfluszt sowohl die Struktur als auch die chemische Zusammensetzung von wachsenden Zellwänden.

Nach Färbung mit ClZnJ. konnten wir in den Zellwänden von Wurzelhaaren ein Dichroïsmus constatieren, der uns instruiert über die Anordnung der Mizellen in diesen Wänden. In normalen Wänden des zylindrischen, nicht mehr wachsenden Teil des Wurzelhaares ist die Mehrzahl der Mizellen parallel an die Längsachse der Zelle gerichtet.

Unter Einflusz von Colchizin zeigen Wurzelhaare abnormale Wachstumserscheinungen, n.l. Wandverdickungen der Spitze, Aufblähungen der Spitze, Verzweigungen u.s.w. Die Mizellen werden hierbei unregelmässig abgesetzt, entweder in die Richtung lotrecht auf die Längsachse der Zelle, oder schräg, oder ungerichtet.

Eine chemische Analyse von in Colchizin gewachsenen Wurzeln von *Hyacinthus* zeigte einen geringeren Zellulose- und Pentosangehalt und einen erhöhten Pektin- und Ligningehalt.

Auf theoretischen Gründen müssen wir annehmen, dass Colchizin eine Änderung der mechanischen Eigenschaften der Zellwand verursacht, nämlich eine Abnahme der Elastizität und eine Zunahme der Plastizität.

Summary.

Colchicin influences the structure as well as the chemical composition of growing cellwalls.

After staining cellwalls of roothairs with ClZnJ, a dichroïsme instructs us about the position of the micells in the walls. In normal roothairs this position is parallel to the longer dimension of the cell, at least in the cylindrical part of the cell, that shows no more growth in length.

Colchicin causes various sorts of growth abnormalities, viz. a thickening of the cellwall of the tip, a swelling of the tip, branchening e.c. In these cases the micelles are deposed irregularly, viz. in the direction perpendicular to the length axis of the cell, or in a spiral or totally indirected. Chemical analysis of *Hyacinthus* roots, grown in colchicin solutions, shows that the quantity of cellulose and pentosanes has decreased as compared with normal roots; pectin and lignin have increased.

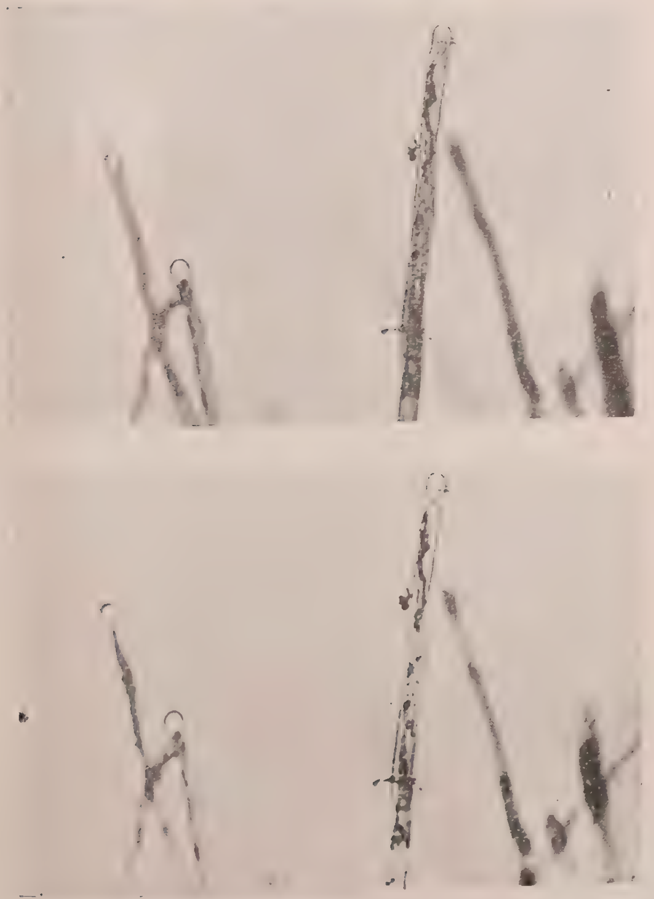
On theoretical grounds we assume, that colchicin influences the mechanical properties of cellwalls, decreasing the elasticity and increasing the plasticity.

Résumé.

On peut constater deux sortes d'influences de colchicine sur les parois cellulaires croissantes: sur la structure et sur la constitution chimique. Après avoir teint les parois de radicelles avec chlorezincjode, nous pouvons voir une dichroïsme, qui nous instruit sur la position des micelles dans les parois. Dans une radicelle normale les micelles se trouvent dans la direction parallelle à l'axe de la cellule.

La colchicine fait naître des abnormalités de croissance; c'est à dire des épaississements de la paroi de la pointe, des gonflements de la pointe, des ramifications etc. Dans ces cas les micelles se trouvent disposés irrégulièrement; en partie dans la direction transversale, dans une direction oblique, ou bien sans direction définie. Une analyse chimique des racines de *Hyacinthe* nous montre, que la quantité de cellulose et de pentosanes est diminuée en comparaison de la quantité dans les racines normales; la quantité de pectine et de lignine est augmentée.

Foto 1.
Wortels van *Hyacinthus orientalis*, groeiend in een 0.025 % oplossing van colchicine. De wortels vertoonen een opzwelling van de groeizône.



A.

B.

Foto 2.
Normaal wortelhaar van *Lepidium sativum*, gekleurd met ClZnJ en gephotografeerd met gepolariseerd licht. Bij foto *A* is de trillingsrichting van het licht 90° gedraaid ten opzichte van de trillingsrichting bij foto *B*. In *A* is het haar donker gekleurd, in *B* licht gekleurd.

A. *B.*

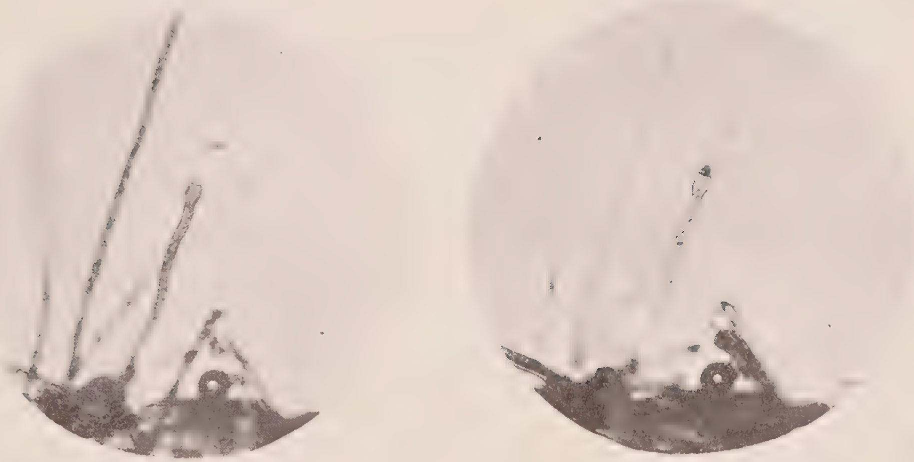


Foto 3.
Wortelhaar van *Lepidium sativum*, gegroeid in colchicine-oplossing 0,5 %, gekleurd met ClZnJ en gefotografeerd met gepolariseerd licht met trillingsrichtingen loodrecht op elkaar. De wandverdikking aan den top vertoont de tegengestelde kleur als de steel.
A. donkere steel, lichte wandverdikking;
B. lichte steel, donkere wandverdikking.

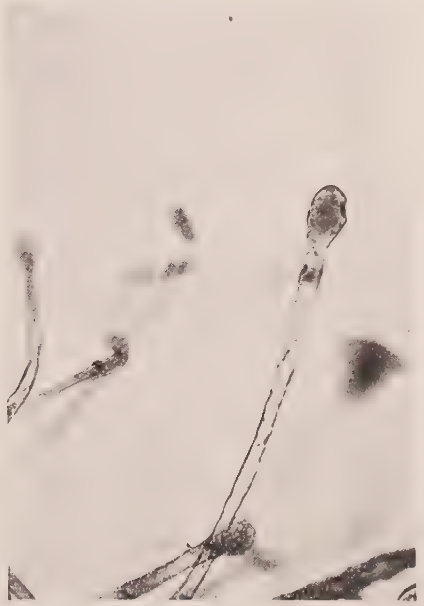


Foto 4.
Wortelhaar van *Lepidium sativum*, gegroeid in colchicine-oplossing 0.5 %, gekleurd met ClZnJ en gefotografeerd met gepolariseerd licht. De opzwelling aan den top is tegengesteld gekleurd aan den steel, n.l. steel licht, opzwelling donker.

A. *B*

Foto 5.
Wortelhaar van *Lepidium sativum*, gegroeid in colchicine-oplossing 0.5 %, gekleurd met ClZnJ en gefotografeerd met gepolariseerd licht met trillingsrichtingen loodrecht op elkaar. Opzwelling aan den top. Deze is tegengesteld gekleurd als de steel. Nl. foto A — donkere steel, lichtere opzwelling, foto B — lichte steel, donkerder opzwelling.

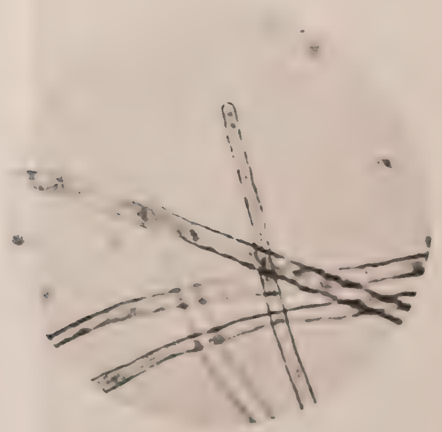


Foto 6.
Wortelhaar van *Avena sativa*, gegroeid in vochtige lucht tegen poreuze plaat, doordrenkt met colchicine-oplossing 0.25 %, gekleurd met ClZnJ en gefotografeerd met gepolariseerd licht; men ziet een schuine streping in den wand.

A cause de considerations théorétiques nous acceptons que l'influence de colchicine se manifeste dans un changement des qualités méchaniques des parois, c.-à-d. une diminution de l'élasticité et une augmentation de la plasticité.

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Botany. — *A new micro method for X-ray structure investigations.* By D. KREGER. (Communicated by Prof. G. VAN ITERSON JR.)

(Communicated at the meeting of October 27, 1945.)

Our knowledge concerning the submicroscopical structure of cells and tissues and other small objects of biological interest has been advanced very much by means of the polarisation microscope.

It would be of great importance if the possibility was given to get X-ray diagrams of these objects too, since the informations about the submicroscopical structure delivered by this method are obtained in a more direct way and allow of further reaching conclusions [1]). The difficulties, however, to get a good X-ray diagram increase as the dimensions of the object decrease and this is the reason why many objects of biological interest remained uninvestigated in this respect.

Investigations on plant waxes, started in September 1941 at the X-ray Institute of the University College of Technology in Delft, also clearly demonstrated the want of a micro-method for X-ray investigations.

From these points of view it seemed worth trying to design a micro-method to enlarge the possibilities for biological investigations at the X-ray Institute.

In 1931 O. KRATKY [2]) published a method for investigating microscopical crystals by means of X-rays, his so called "Mikrokonvergenzmethode". He succeeded in obtaining a diagram of a $KClO_3$ crystal of 12 μ long and 6 μ thick. Although valuable for microscopic monocrystals this method has its drawbacks for our purposes and we decided on trying to find other ways.

The most important problems in designing a micro-diffraction camera are:

1. The construction of so narrow a diaphragm as is required to get an X-ray beam of microscopical dimensions and of which material to make it;
2. The screening of the radiation diffracted at the edge of this diaphragm;
3. The mounting of the small object in the beam;
4. An estimate of the limits of the method and a discussion of the factors by which they are determined.

From October 1942 to September 1944 (with interruptions due to the war) we succeeded in working out a method in which all these problems

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1) For advantages and drawbacks of the two methods of investigation see: W. J. SCHMIDT, Polarisationsoptische Analyse des submikroskopischen Baues von Zellen und Geweben. ABDERHALDEN, Handbuch der biologischen Arbeitsmethoden Abt. V, Teil 10, 440—444 (1934).

2) Z. Krist. 76, 261 (1931).

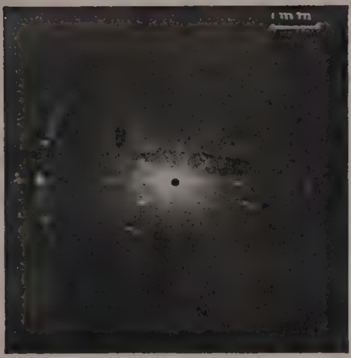

Fig. 1.

Diagram of tungsten thread, 12 μ thick, taken with diaphragm of 6 μ diameter and screened with screening cup of 16 μ diameter and 100 μ length. Exposure time 5 hours at 25 kV and 100 mA, with Cu anode. Distance object-film about 1,5 mm. As the film is coated on both sides and the interference beams pass the film at an oblique angle, each spot is double.

Fig. 2.

Diagram of a chitin hair of a gnat, 12 μ thick. Diaphragm 20 μ, screening cup 40 × 250 μ. Exposure 8 h at 25 kV, 100 mA, with Cu anode. Distance object-film about 3 mm. Film coated on both sides.

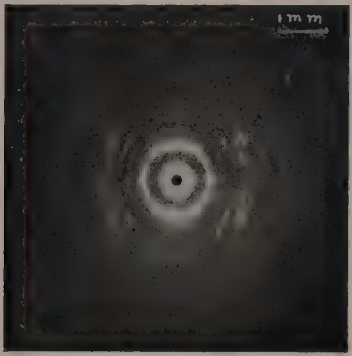

Fig. 3.

Diagram of sugar-cane wax. The wax grows on the cane stalk in a thin layer consisting of small rods, ± 4 μ thick and ± 80 μ long, which are placed perpendicular to the surface of the stalk. The diagram is taken of a packet of these rods of ± 50 μ thick, the X-ray beam passing the rods in a direction perpendicular to their length. Diaphragm 50 μ, screening cup 105 × 400 μ. Exposure 2,5 h at 25 kV, 100 mA, Cu anode. Distance object-film about 1,9 mm. Film coated on *one* side.

Fig. 4.
Diagram of fig. 3, natural size.

are solved. We used diaphragms of a conical form and of leadglass. KRATKY used an elongated slit of gold. Above KRATKY's method the method has the advantage that the dimensions of the diaphragm, screening cup and distance of the object to the film can be altered in correspondence with the requirements of the object and further, that reflections with a small glancing angle are not lost in the central opening in the film or in the blackening in the aequator of the diagram.

The method enables us to make X-ray beams with a diameter of 5 μ and less. For organic (not for inorganic) specimens of this dimension the exposure time, however, becomes too long. Therefore the inferior limit for the size of organic specimens may be put equal to 10 μ and likewise for the corresponding diaphragms. In this case an exposure time of 20 to 50 hours is assumed, with a rotating anode at 25 kV and 100 mA. Up to now, however, there was no opportunity to make a diagram of an organic object with such a narrow diaphragm and requiring an exposure time of the above mentioned length in consequence of lack of energy and other difficulties due to the war.

The best diagrams obtained hitherto are shown in the next figures. They are enlarged circa 8 times.

The supposition that an exposure time of 20 to 50 hours for an organic object and 10 μ diaphragm will be sufficient is based on the exposure time required for the diagram of fig. 2.

More details about method and diagrams will be supplied elsewhere.

Delft, August 1945.

Institute for X-ray Investigation
University College of Technology.

Palæontology. — *Upper Cretaceous Smaller Foraminifera from Buton (D. E. I.).* By F. G. KEIJZER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of October 27, 1945.)

During the study of the microfauna of asphalt-bearing tertiary marls from the island of Buton (SE of Celebes in the Duch East Indies) reworked upper cretaceous smaller *foraminifera* were found in one of the samples. Generally these are almost pure *Globigerina-Globorotalia*-marls with the faunal assemblage very common in the Neogene of the Indo-pacific region. One sample, however, contains many sharply angular white and grey pieces of limestone and grey and black pieces of chert, resembling in alle respects, macroscopically as well as in thin-section, the cretaceous *Globotruncana*-limestones from the island present in the collection of the Geological Institute at Utrecht. In this sample many upper cretaceous smaller *foraminifera* are mixed with the tertiary assemblage. The cretaceous forms differ markedly from the beautifully preserved tertiary *foraminifera* in their dull grey colour and no mistake can be made in their picking from the sample. The following upper cretaceous forms could be identified:

Cristallaria (Robulus) cf. inornata (D'ORB)
Nodosaria sp
**Gümbelina camagueyana* VAN RAADSHOVEN
**G. costulata* CUSHMAN
**G. excolata* CUSHMAN
**G. globulosa* (EHRENBERG)
**G. nuttalli* VOORWIJK
**G. plummerae* LOETTERLE
G. pseudotessera CUSHMAN
**G. reussi* CUSHMAN
**G. striata* (EHRENBERG)
**Pseudotextularia varians* RZEHAK
**Planoglobulina acervulinoides* (EGGER)
**Ventilabrella carseyi* PLUMMER
**Bolivinoides seranensis* VAN DER SLUIS
Pulvinulinella culter (PARKER & JONES)
**Globigerina bulloides* D'ORBIGNY
**G. cretacea* D'ORBIGNY
**G. pseudotriloba* WHITE
**Globigerinella aspera* (EHRENBERG)
**Globorotalia crassaformis* (GALLOWAY & WISSLER)
**G. membranacea* (EHRENBERG)
**G. velascoensis* (CUSHMAN)
**Globotruncana arca* (CUSHMAN)

G. calcarata CUSHMAN
G. canaliculata (REUSS)
G. canaliculata (REUSS), *var. ventricosa* WHITE
G. convexa SANDIDGE, *var. multicamerata* VAN DER SLUIS
Anomalina bentonensis MORROW
A. sp.

In this list species marked by an asterisk have also been found by VAN DER SLUIS in the Upper Cretaceous of Seran (D. E. I.). The resemblance with that fauna is so great, that both faunae may safely be regarded as synchronic (i. e. rather high in the Upper Cretaceous).

The cretaceous rocks are known already for a long time through the investigations made on the island by the governmental "Dienst van den Mijnbouw". The knowledge has been summarized by HETZEL [1]) in the final report on the asphaltbearing rocks on Boeton. The cretaceous series consists of white, grey and rosy, often porcelain-like dense-textured limestones, often cut by veinlets of calcite, and containing the typical assemblage of

Gümbelina globulosa (EHRENBERG)
Globigerina cretacea D'ORBIGNY
Globotruncana canaliculata (REUSS).

This type of limestone is very common in the eastern part of the East Indian Archipelago. Characteristic for the formation are layers and concretions of reddish and yellow chert, sometimes rich in *radiolaria*.

The dense-textured limestones are mentioned to occur sometimes in alternation with white or light-grey marly limestones with locally also layers of chert. In contrast with the first type these limestones contain abundantly *Globigerinidae*, but no *Globotruncana*. This may be interpreted as due to changes in facial conditions, but it remains also possible that the marly *Globigerina*-limestones belong the another stratigraphic unit than the *Globotruncana*-limestones. The last possibility leads to the assumption of tectonical complications in the sections relating to the matter in question. HETZEL's report, however, is too summary to make a further discussion on this point possible, but the cretaceous series is reported to be strongly folded and sheared. Tectonical complications in Boeton have also been suggested by WANNER (*2*) for a succession of strata where, according to BOTHÉ (3), upper jurassic limestones are overlain as well as underlain by *Globotruncana* ("*Rosalina*")-limestones (Cretaceous).

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2. J. WANNER, Gesteinsbildende Foraminiferen aus Malm und Unterkreide des östlichen Ostindischen Archipels. *Pal. Ztschr.* Bd. 22, p. 75—99 (1940).
3. A. C. D. BOTHÉ, Voorloopige mededeeling betreffende de geologie van Zuid-Oost-Celebes. *De Mijningenieur* VIII, p. 97—03 (1927).

Palæontology. — *Corals from the Upper Miocene of Tjisande, Java.* By J. H. F. UMBGROVE. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of October 27, 1945.)

In his description of sheet 54 of the geological map of Java 1 : 100.000 (Ao. 1935) Dr. W. H. HETZEL mentioned the occurence of a reef limestone along the Tjisande, North of Lurahgung (Central Java). Reef limestones occur rather abundantly in the form of lenses in the upper part of the so-called Halangbeds. Originally they must have grown up as patch reefs in the same way as the present reefs in the bay of Batavia and the Spermonde Archipelago. Gradually they were buried by the accumulation of muddy sediments.

The marly limestone which is overlying the reef limestone at Tjisande contains worn-off fragments of Cycloclypus and Lepidocyclina derived from older strata. Moreover in the same beds a tooth was found of a rhinoceros, Aceratherium boschi Von Koenigswald (the oldest remain of a land vertebrate so far known from Java). It seems therefore that the limestone of Tjisande belongs to the Lepidocyclina-free part of the Neogene and should be reckoned to the uppermost miocene, tertiary g. As a matter of fact this conclusion is supported by the results of my examination of the corals from the Tjisande limestone.

The collection consists of 21 different corals and the species of 15 of these could be identified. Among these 15 species 7, i.e. 46,6 percent are species which are still living among the recent fauna's of the Indo-Pacific. This percentage figure shows the Tjisande limestone to be undoubtedly older than the coral-bearing localities in the pliocene Sonde beds of Java that belong to the Tertiary h. On the other hand it proves the Tjisande reef to be younger than coral fauna's known from the Lepidocyclina-bearing Upper Miocene (Tertiary f) of the East Indies. A very interesting lower-pliocene coral fauna of Prupuk in Central Java — being the largest suite of tertiary corals so far collected from one single locality in the East Indies — contains 54 percent recent species.

Many coral species of Tjisande were found also in the coral reef of Prupuk, viz. 71,4 percent. Most of the specimens from Tjisane could even be identified only by comparing them to the much better preserved material from Prupuk. It seems, therefore, that geologically speaking the Tjisande reef is only slightly older than the fossil reef of Prupuk and might belong either to the lowermost Pliocene or to the uppermost Miocene. This conclusion is in agreement with HETZEL's statement that the reef lenses of Tjisande occur in the upper part of the Halang series, i.e. in the transition beds towards the lower pliocene Kumbang beds.

The following table summarizes the identified corals from Tjisande and their distribution in the upper miocene Tjilanang beds, the lower pliocene Prupuk reef and the recent fauna.

	Corals from Tjisande	Tjilanang	Prupuk	Recent
1	Seriatopora spec.		+	
2	Stylophora pistillata Esper	+		+
3	Antillophyllia constricta (Brügg.)			+
4	Lithophyllia grandissima Felix		+	
5	Galaxea clavus (Dana)		+	+
6	Galaxea spec. 1.		+	
7	Favites spec. 1		+	
8	Coelastrea rectangularis nov. spec.			
9	Coeloria cf. daedalea (Forsk.)		+	+
10	Merulina ampliata (Ell. et Sol.)	+	+	+
11	Lobophyllia corymbosa (Forsk.)		+	+
12	Lobophyllia costata (Dana)		+	+
13	Diploastrea heliopora (Lam)		+	+
14	Fungia pseudo echinata Gerth		+	
15	Fungia spec. 1.		+	
16	Pachyseris curvata Martin	+	+	
17	Cyathoseris lophiophora Felix		+	
18	Cyathoseris cf. crassilamellata Gerth		+	
19	Pavona micrommata Felix			
20	Tubipora spec.			
21	Isis spec.			
		3	15	8

Species 1, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18 will be treated at great length in a paper on the Prupuk reef which is ready for publication. This is the more reasonable as no 6 and 7 belong to new species, which are represented by much better perserved specimens in the collection from Prupuk.

No 3 was already mentioned by me when I described Antillophyllia constricta in my paper on the corals from Talaud.

So. I may restrict myself here in publishing only a description of the new species Coelastrea rectangularis (no 8) and in giving some comments on nos. 2, 4, 19, 20 and 21.

Stylophora pistillata Esper

1880. Stylophora digitata (Pallas), K. MARTIN, Die Tertiärschichten von Java, p. 135, plate 24, fig. 9 and 10.

1912. „ „ „ J. FELIX, New Guinea, Ber. K. Sächs. Akad. d. Wissensch. 64, p. 443.

1913. „ cf. „ „ J. FELIX, Trinil, Palaeontogr. 60, p. 360.

1915.	,,	pistillata Esper,	J. FELIX, Palaeont. von Timor II, p 40.
1921.	,,	,, ,,	J. FELIX, Borneo, Palaeont. von Timor IX, p. 52.
1922.	,,	digitata Pallas,	H. GERTH, Java, Samml. Geolog. Reichsmus. Leiden I.2, p. 420.
1923.	,,	pistillata (Esper),	H. GERTH, Borneo, Samml. Geolog. Reichsmus., Leiden I,X, p. 96.
1925.	,,	,, ,,	H. GERTH, Nias, Leidsche Geol. Med. I, p. 32
1926.	,,	,, ,,	J. H. F. UMBGROVE, Sumatra, Wetensch. Med. Nr. 4, p. 41.
1929.	,,	,, ,,	J. H. F. UMBGROVE, Java, Tijdschr. Kon. Ned. Aardr. Gen. 46, p. 11.
1929.	,,	,, ,,	J. H. F. UMBGROVE, Borneo, Wetensch. Meded. Nr. 9, p. 63, fig. 2 and plate 3, fig. 40—42.

For synonymy and distribution of recent species see UMBGROVE, Zoolog. Mededeel. Vol. 22, p. 23 and 274.

One well preserved specimen showing a striking resemblance to the coral figured by BEDOT (Madréporaires d'Amboine) in his plate 5 fig. 6.

Distribution: Upper Miocene: Borneo, Java; Pliocene: Java; Plio-Pleistocene: Timor, Sumatra, Nias; Recent: Indo-Pacific.

Lithophyllia grandissima Felix

1921.	Lithophyllia grandissima Felix,	Palaeontologie von Timor IX, p. 24, Pl. 3, fig. 5, 5a.
1923.	,, ,, ,,	H. GERTH, Anthoz. von Borneo Samml. Geol. Reichsmus., Leiden X, p. 64.

There is one specimen, which has well preserved septa and costae, both with broad-based spines as was described in detail by FELIX. A seciton through the coral shows a strongly developed vesicular endotheca. Diameter of calice 4 to 7,5 cm. Calicular margin rounded. Septa in 5 complete cycles and a 6th cycle incomplete. FELIX has not given a description of the columella. GERTH, however, mentions a specimen having a narrow columella, 3 cm in length. The columella of the specimen from Tjisande is not in a good state of preservation, but it can hardly have been more than a spongy columniform mass with a circular diameter.

There are, however, two more specimens from Tjisande, and, although these are badly damaged, one of them shows a columnella of very irregular shape, divided as it where in two separate centra.

Two fragments from the Prupuk reef, apparently belong to the same

species. One of them has a columnella consisting of a loose spongy structure, circular in diameter. It appears, therefore, that the shape of the columella is rather variable in this species.

Distribution: British Borneo from a locality of unknown age (Felix). East Borneo from the surroundings of Bontang, amidst a fauna, which Gerth considers to be of Pliocene age.

Coelastraea rectangularis nova species, Fig. 1.

A lone fragment (holotype) of a well preserved coral from Tjisande shows polygonal to rectangular calices, which seem very characteristic. Calices deep, up to 6 mm. Calices fused by their thin walls. Diameter of calices 5 by 5 mm up to 5 by 11 mm. In the larger calices 12 steeply dipping septa nearly reach the calicular centre; between these 12 thinner and slightly shorter septa occur and a few rudimentary ones of the fifth order. In smaller calices only three orders are complete and a few of the fourth order may occur. Septa, when well preserved, show broad-based dentations. Septal faces smooth. Septa usually continuous and exsert over the intercorallite wall, exceptionally alternating. Columella deep, loose; a small trabecular structure, not visible in undamaged calices. Corallum vesicular, dissepimental. Lower side lobed and costate.

Pavona micrommata (Felix)

1912.	Stephanocoenia intersepta (Esper),	J. Felix, New Guinea, Ber. d. Sächs. Akad. d. Wissensch. 64, p. 444.
1913.	Siderastraea micrommata Felix,	Trinil, Palaeontogr. 60, p. 335. fig. 3.
1915.	„ „ „	Palaeont. von Timor II, p. 34.
1924.	Stephanocoenia intersepta (Esper),	J. H. F. Umbgrove, Geol. Results of Explor in Ceram II, 1, p. 11.
1926.	Siderastraea micrommata Felix,	J. H. F. Umbgrove, Sumatra, Wetensch. Meded. Nr. 4, p. 43.

Felix has given a good description. The surface of the colony is only locally well preserved, for the greater part of the surface layers are rather worn off, the deeper structure appearing to view. The specimen is irregular noduliform, 115 mm long, 70 mm broad. Pavona duerdeni Vaughan (1917) is perhaps an allied form among the recent coral fauna, but it has a compressed, often lamellate columella and a deeper calicular fossa.

Distribution: Plio-Pleistocene (?): New-Guinea, Ceram, Sumatra; Pliocene (?): Trinil (Java), Timor.

Tubipora spec., fig. 2 and 3, nat. size.

Two fragmentary specimens. Diameter of the corallites varying from 1 to 2 mm, usually 1,5 mm. Four corallites to 1 centimeter rarely more. Horizontal laminae 8 up to 10 mm apart (fig. 3). The specimens probably belong to a still living species, perhaps Tubipora musica Linn., but an identification of the species is not possible because of the bad state of preservation of the fossils. Tubipora rubiola Quoy et Gaim was mentioned by FELIX from the Pliocene (or Pleistocene) of Timor (Pal. von Timor VIII, 1920, p. 25).

Isis spec. Fig. 4, 5 and 6, nat. size.

Perhaps a specialist in this group might be able to identify a calcareous body and a fragment of the basal part of Isis from Tjisande.

I refrain, however, from constituting a specific name on so scanty material. The basal portion is an incrustation, showing narrowly waved ridges and furrows (fig. 6). The branching of the calcareous body (fig. 4) shows it to belong to the genus Isis and not to Mopsea. Its length is 65 mm; diameter oval 15 $\times$ 7 mm. One condylus is well preserved (fig. 5), the other end is broken off. Ridges and furrows straight and very distinct. No granules on the ridges. The species seems different from Isis cf. polyacantha, which was described by FELIX from the Plio-Pleistoncene of Timor. According to my opinion Isis danae, Isis elongata and Isis compressa, described by DUNCAN from the miocene Gai series of Sind (Palaeont. Indica 1886) all belong to Isis polyacantha Steenstrup. The calcareous body from Tjisande seems also different from the Isidac described by DUNCAN from tertiary deposits of New Zealand (Quart. Journ. Geol. Soc. 31, 1875) and by TENNISON Woods from New Zealand (Palaeont. of New Zealand IV, 1880).

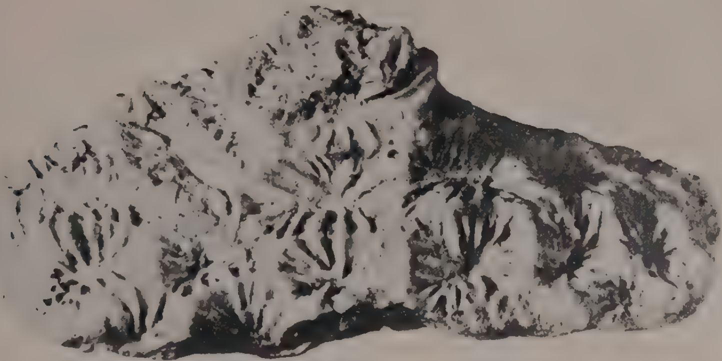

Fig. 1.
Coelastrea rectangularis nova species. $\times$ 2.

Fig. 2 and 3.
Tubipora spec. Nat. size.

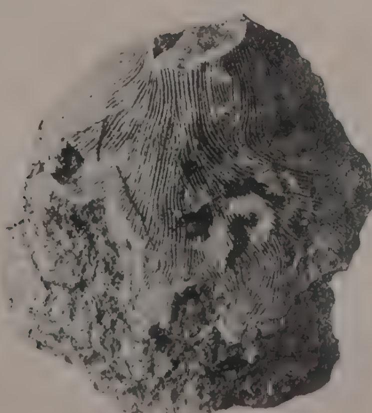

Fig. 4, 5 and 6.
Isis spec. Nat. size.

Anatomy. — *The nucleus of* BELLONCI *and adjacent cell groups in Selachians. I. The ganglion habenulae and its connections.* By J. L. ADDENS. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of October 27, 1945.)

In the first (1938) of this series of contributions on the nucleus of BELLONCI and its environment the presence of the nucleus in question, described for the first time in 1888 [1]) by BELLONCI in the frog, was dealt with in Reptiles and Mammals, in which groups it probably is of general occurrence. In a second paper (1940) I demonstrated its presence in Birds. It thus remained to be investigated whether the nucleus is already found in Cyclostomes and Fishes. In a communication in Dutch, appeared in the "Verslagen" of this Academy in two parts (1943, '43a), this question, as far as concerns Cyclostomes, was answered in the affirmative. In Petromyzonts as well as in Myxinoids a well-developed nucleus of this kind could be demonstrated.

Thus only Fishes are wanting to a complete survey of the occurrence of the nucleus of BELLONCI throughout Vertebrates.

As mentioned in the introduction to the paper on Cyclostomes I found only one statement in literature about our nucleus in Fishes, viz., in Teleosts (JANSEN, 1929). I shall enter upon it below. About the lower groups of Fishes no data are known to me.

In the present paper we shall restrict ourselves to Selachians, of each of whose three groups, Sharks, Rays and Holocephali, representatives could be investigated. Of the following species series stood at my disposal:

S h a r k s

Notidanidae: Heptanchus cinereus, Hexanchus griseus; *Scylliidae:* Scyllium canicula; *Cetorhinidae:* Selache maxima; *Spinacidae:* Acanthias vulgaris, Centrophorus granulosus, Spinax niger.

R a y s

Rajidae: Raja circularis, R. clavata; *Torpedinidae:* Torpedo marmorata.

H o l o c e p h a l i

Chimaera monstrosa.

Most of these series were alternating WEIGERT-PAL-paracarmine and VAN GIESON series, but among them were some stained for cells only (haematoxylin and lithium-carmine), as also some CAJAL series (of Chimaera monstrosa).

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[1]) Titles of papers cited which are not found in the list of literature of this communication, to follow at the end of the second part, may be looked up in the lists of my previous contributions to the subject (1938, '40, '43a).

The topography and structure of the nuclei concerned, als well as the course of the tracts supplying them or arising in them, in all species investigated, whether Sharks, Rays or Holocephali, are so uniform that the detailed description of one representative suffices. Instead of the usual example for Selachians, viz., Acanthias, Scyllium canicula was chosen for this purpose, because more and better series of this species stood at my disposal. Thus the following description principally is based on a transverse and a sagittal WEIGERT-PAL-paracarmine series of Scyllium canicula. For the cytological relations, moreover, a transverse haematoxylin and a transverse lithium-carmine series of this animal were examined. For the drawings of sections the transverse (figs. 1—4) and the sagittal WEIGERT-PAL-paracarmine series (figs. 5, 6) were used, while the reconstructions of the left and right habenula and their environment (figs. 7, 8) were made after the transverse WEIGERT—PAL-paracarmine series, with the aid, however, of the sagittal one, as in the transverse series no indicator lines were traced.

The ganglia habenulae.

Although it is our chief aim to search for the nucleus of BELLONCI, we better start with the description of the ganglia habenulae (figs.1—8).

The ganglia habenulae, in Selachians, are strongly developed and occupy the foremost half of the epithalamus, whose hindmost half is occupied by the pars intercalaris diencephali (figs. 5, 6). Dorsally the ganglia habenulae are traversed by the commissura habenularum, which likewise is strongly developed and in longitudinal sections, cutting it, of course, transversely, is approximately round (figs. 3—8).

Immediately above the commissure the recessus epiphyseos (Proximalpartie des Pinealorganes of STUDNIČKA, 1905) is situated, a dilatation of the basis of the epiphyseal stalk, which opens into the horse-shoe-shaped subcommissural organ. In the frontally looking concavity of this organ the commissura posterior passes, which on sagittal section is likewise horse-shoe-shaped (figs. 5—8). Also the pars intercalaris diencephali shows this form, being closely applied, on either side of the subcommissural organ, to the commissura posterior. It is no clearly defined structure, being continuous laterally with the pars dorsalis diencephali, from which it is not separated by a sulcus dorsalis.

The curving up, in frontal direction, of all these structures, to which must be added the tractus tecto-habenularis, later to be described, and the aquaeduct, no doubt is due to the strong development, in Selachians, of the optic tectum.

According to the majority of statements in literature, in Selachians, contrary to Cyclostomes, the left ganglion habenulae is larger than the right one. This, however, is not the case. Also in Selachians the right ganglion is the larger one. The difference in size, however, mainly due

to the somewhat more frontal beginning of the right ganglion (figs. 5, 7, 8), is but small, contrary to Cyclostomes, where in Petromyzonts as well as in Myxinoids, the right ganglion surpasses the left several times in size.

While the difference in size between right and left ganglion is but small, their difference in histological structure is considerable. In preparations with myelin staining it strikes immediately that the dorsolateral region of the right ganglion is provided with a dense network of myelinated fibres (figs. 1—3, 6). Likewise it is striking in such preparations that the right fasciculus retroflexus is far more myelinated, and stronger than the left (figs. 1—4). The cells in the meshes of the myelinated network of the right ganglion are considerably larger and lie more loosely arranged than the rest of the cells of the ganglion, which like all cells of the left ganglion show the usual small habenula type and like these lie tightly packed together.

I now proceed to a more detailed description of the ganglia habenulae. As is generally known, these ganglia usually are divided into two, a medial and a lateral, or a dorsal and a ventral nucleus, resp. Already in Cyclostomes this division is met with, viz. into a dorsal and a ventral nucleus, or more exactly an antero-dorsal and a postero-ventral nucleus (cf. ADDENS, 1943, 43a).

In Scyllium, and in Selachians generally, the division into two of the ganglia habenulae is not very pronounced. The character of the cells, which are small and lie tightly packed together, is the same throughout the ganglion, except in the lateral upper corner of the right one, where, as already stated, the cells are larger and lie more loosely arranged. A division into two, however, of both ganglia, is indicated by grooves, though only in their foremost part (figs. 1—3, 6—8).

The two nuclei which so may be distinguished in Scyllium, strictly spoken, ought to be designated as dorsomedial and ventrolateral. The difference, however, in horizontal level is so small that, for convenience, we may speak of a medial and a lateral nucleus. In other species, Acanthias vulgaris, for instance, the two nuclei lie more vertically one above the other, and in Chimaera monstrosa they even do so perpendicularly, so that here must be spoken of a dorsal and a ventral nucleus.

In both ganglia, of which the right, as stated above, extends somewhat more frontally, the lateral nucleus begins first. The medial right nucleus, curving in front of the left ganglion, a little transgresses the median plane (fig. 5).

On the ventricular side the separation between medial and lateral nuclei is formed by the sulcus intrahabenularis internus, which from before backwards gradually becomes more shallow (figs. 1—3, 6—8). On the left side this groove is very short, on the right side, however, it extends over more than half the length of the nucleus (figs. 7, 8). On the left side the separation between medial and lateral nuclei, moreover, is formed by an external groove, the sulcus intrahabenularis externus (fig. 1). This groove,

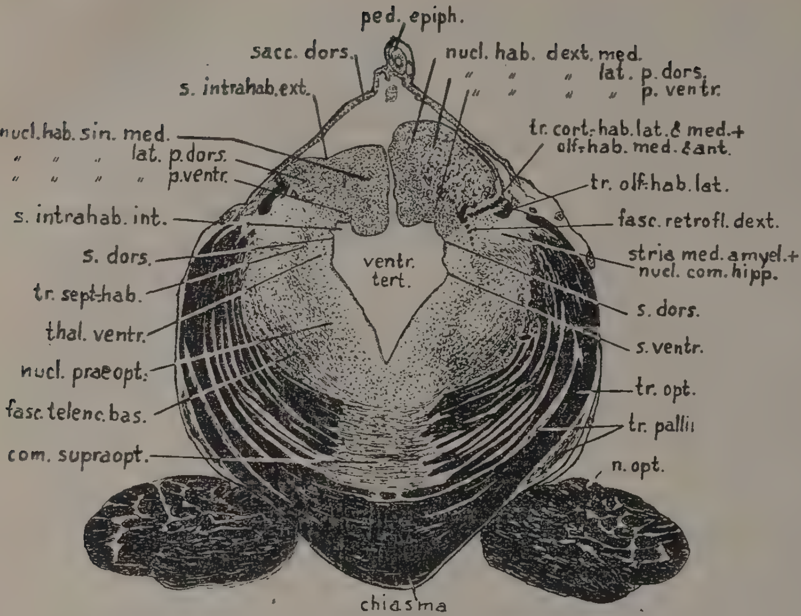


Fig. 1.

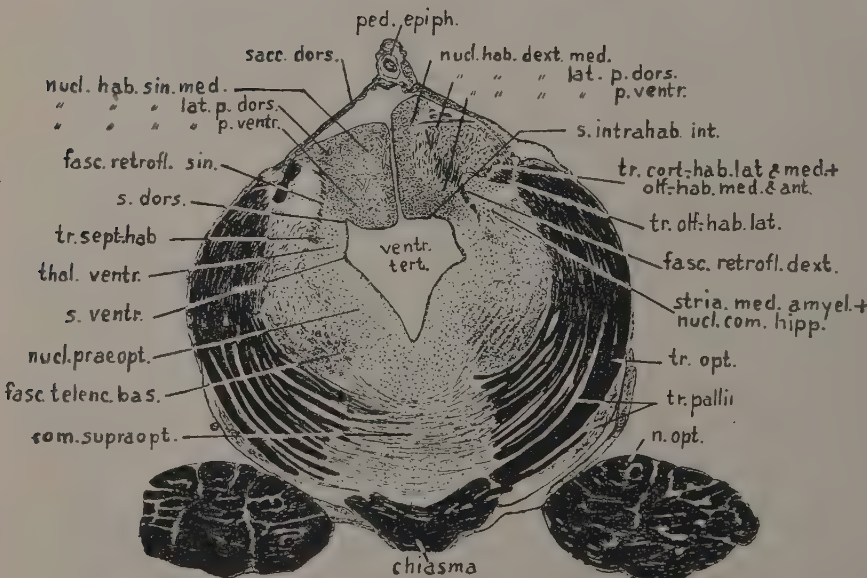


Fig. 2.

Figs. 1—4. *Scyllium canicula.* Cross-sections through ganglia habenulae, nucleus of BELLONCI and nucleus commissurae hippocampi. WEIGERT–PAL-paracarmine. × 18. The position of the sections is indicated in the reconstructions figs. 7 and 8 by interrupted lines.

which had not been described before, we also found in the right ganglion habenulae in both transverse series of Petromyzon fluviatilis examined by us (1943, '43a). Here an internal groove was absent. Other investigators, however, in Petromyzonts, report only an internal groove in the right ganglion. On the left side there are here neither internal, nor external intrahabenular grooves. In Myxinoids they are absent on both sides.

It follows from the foregoing that in Scyllium there is no sharp separation of the ganglia habenulae into a medial and a lateral nucleus. In the foremost part of the left ganglion the line connecting the external and internal intrahabenular sulci may be regarded as boundary. It also is but natural to reckon the large-celled dorsolateral upper corner of the right nucleus with the lateral nucleus. Thus we shall call this cell group pars dorsalis of the lateral nucleus. The small-celled part of the right ganglion situated ventral to the pars dorsalis and outside the sulcus intrahabenularis internus, may then be distinguished as pars ventralis of the lateral nucleus.

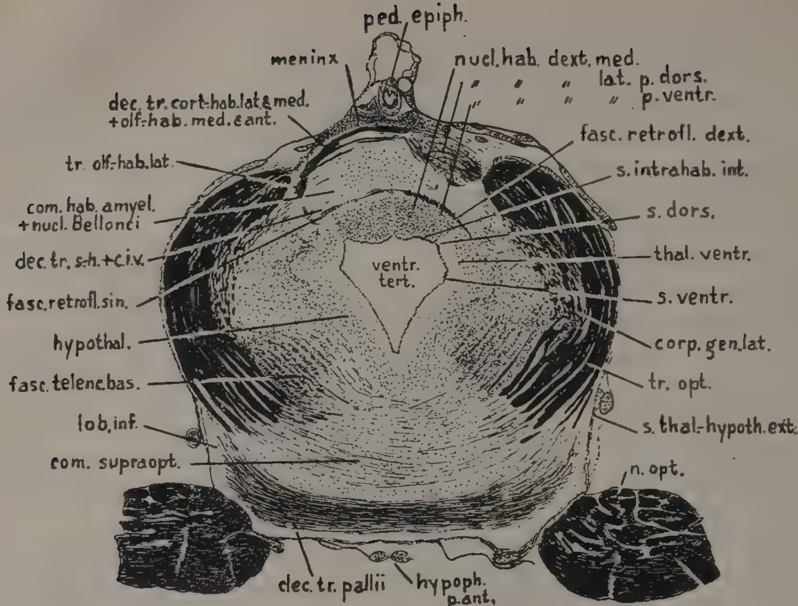
As aforesaid the large-celled part of the right ganglion is densely filled with myelinated fibres. The corresponding region of the left ganglion, whose cells are of the same small type as in the rest of the ganglion and like these lie tightly packed together, at first sight seems to be devoid of myelinated fibres. On closer inspection, however, also here some scattered myelinated fibres are found, and thus also the lateral left nucleus may be divided into a pars dorsalis and a pars ventralis.

The myelinated network in the pars dorsalis of the lateral right nucleus for the most part is formed by the diverging and splitting up of fibres of the commissura habenularum. But also the emerging fibres of the fasciculus retroflexus considerably contribute to its formation.

The scattered myelinated fibres in the pars dorsalis of the left lateral nucleus likewise partly are endings of fibres of the commissura habenularum and partly emerging fibres of the fasciculus retroflexus. It is natural to suppose that the latter, like the myelinated fibres of the right fascicle, arise from larger cells. I did not succeed, however, in finding in the left ganglion among the great mass of small cells, closely lying together, some scattered cells of a larger type.

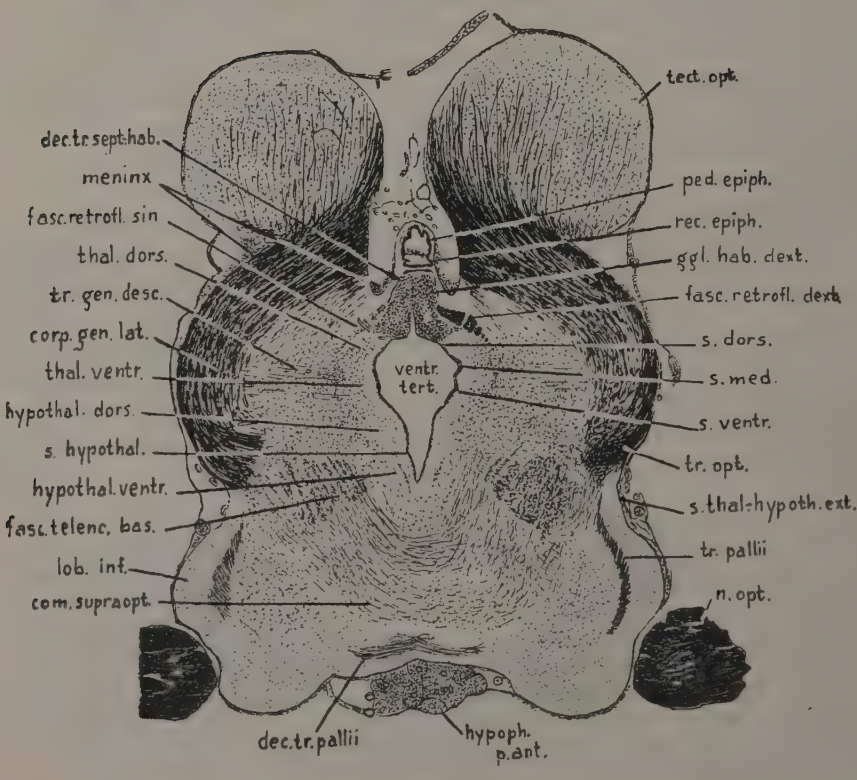
As mentioned above, according to the majority of statements in literature, the left ganglion habenulae, in Selachians, is the larger and gives rise

Fig. 1. Through foremost part of ganglia habenulae, and nucleus commissurae hippocampi.
Fig. 2. Through ganglia habenulae a little in front of commissura habenularum.
Fig. 3. Through ganglia habenulae, commissura habenularum and fused nuclei of BELLONCI.
Fig. 4. Through hind end of ganglia habenulae and commissura habenularum.
dec.tr.s.-h. + c.i.v., decussation of tractus septo-habenularis (according to JOHNSTON's interpretation), and of tractus olfacto-habenularis medialis; tr.gen.desc., tractus geniculatus descendens (not yet described for Selachians). To the label tr.cort.-hab. lat & med., etc., must be added: + amygd.- & strio-hab.



ped. epiph.
meninx
nucl. hab. dext. med.
" " " lat. p. dors.
" " " " p. ventr.
dec. tr. cort-hab. lat. & med. + olf-hab. med. e ant.
fasc. retrofl. dext.
tr. olf-hab. lat.
s. intrahab. int.
com. hab. amyel. + nucl. Bellonci
s. dors.
dec. tr. s-h. + c. i. v.
thal. ventr.
ventr. tert.
fasc. retrofl. sin.
s. ventr.
hypothal.
corp. gen. lat.
fasc. telenc. bas.
tr. opt.
lob. inf.
s. thal-hypoth. ext.
com. supraopt.
n. opt.
dec. tr. pallii
hypoph. p. ant.

Fig. 3.



tect. opt.
dec. tr. sept-hab.
meninx
ped. epiph.
fasc. retrofl. sin
rec. epiph.
thal. dors.
ggl. hab. dext.
tr. gen. desc.
fasc. retrofl. dext.
corp. gen. lat.
s. dors.
thal. ventr.
ventr. tert.
s. med.
hypothal. dors.
s. ventr.
s. hypothal.
tr. opt.
hypothal. ventr.
s. thal-hypoth. ext.
fasc. telenc. bas.
tr. pallii
lob. inf.
n. opt.
com. supraopt.
dec. tr. pallii
hypoph. p. ant.

Fig. 4.

to the myelinated fasciculus retroflexus. How this error has arisen, I am not prepared to say. EDINGER (1892) was the first to indicate a difference in size between right and left ganglion, without saying, however, which is the larger. From the smaller ganglion, according to him, a smaller and unmyelinated fasciculus retroflexus arises. B. HALLER (1898) correctly states that it is the right ganglion which is the larger and gives rise to the stronger fasciculus retroflexus. KAPPERS (1906) and STERZI (1909) state the same, adding that the right bundle is more myelinated. The more recent investigators, however, all of them are of the reverse opinion, and since also the recent text-books have it [1]), this may be said to be the prevalent, or even the generally accepted opinion. That it is not correct, however, was clear from the series enumerated above, belonging to representatives of all the three groups of Selachians. Of these series it was known with reasonable certainty which were the right and left sides. To check this in dissections was not possible, the distance the right ganglion extends more frontally than the left being too small for that.

The fasciculus retroflexus.

The chief efferent tract of the ganglion habenulae, the fasciculus retroflexus, in Selachians is made up of myelinated und unmyelinated fibres. The right and left bundle are asymmetrically developed, in that the right bundle contains far more myelinated fibres (figs. 1—4). So the latter is larger, and more conspicuous, especially in myelin stained preparations.

The right bundle thus presenting the clearer pictures, we start our description of the structure and origin of the fasciculus retroflexus with this. Its myelinated fibres arise exclusively from the large-celled part of the lateral nucleus. They begin to emerge already at the front end of the latter, leaving it ventrolaterally and applying themselves to the medial side of the stria medullaris (fig. 1). Coursing backward the bundle is gradually reinforced medially by fibres arising more caudally (fig. 2). Thus a flat band of myelinated fibres is formed at the medial side of the stria medullaris. The fibres from the hindmost part of the nucleus, which is situated above the commissura habenularum (fig. 3), first run forward, but immediately in front of the commissure bend down and join the bundle medially. The fasciculus retroflexus, now completely formed as to its myelinated part, then courses caudally, wedged in between the commissura habenularum and the small-celled parts of the ganglion (fig. 3). At first a broad, flat band (figs. 3, 4), the bundle, proceeding caudalward, gradually becomes narrower and higher, and splits up into a number of small bundles, which on the side of the ventricle are highest. At last

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[1]) Also BECCARI (1943) mentions this in his text, but in the explanation of fig. 249, representing a cross-section through the habenula of Scyllium canicula, he says: „in Scyllium non esiste differenza di larghezza fra i due nuclei; ma, in senso antero-posteriore, quello sinistro è più corto di quello destro".

the bundle, coursing caudally beside the third ventricle and aquaeduct, is nearly round.

Besides the myelinated fibres just described, there run in the right fasciculus retroflexus a great number of unmyelinated fibres, arising from the small-celled parts of the ganglion. They leave this at its caudal end, where in WEIGERT-PAL-paracarmine preparations reddish spots appear in the black bundles of myelinated fibres composing the fascicle.

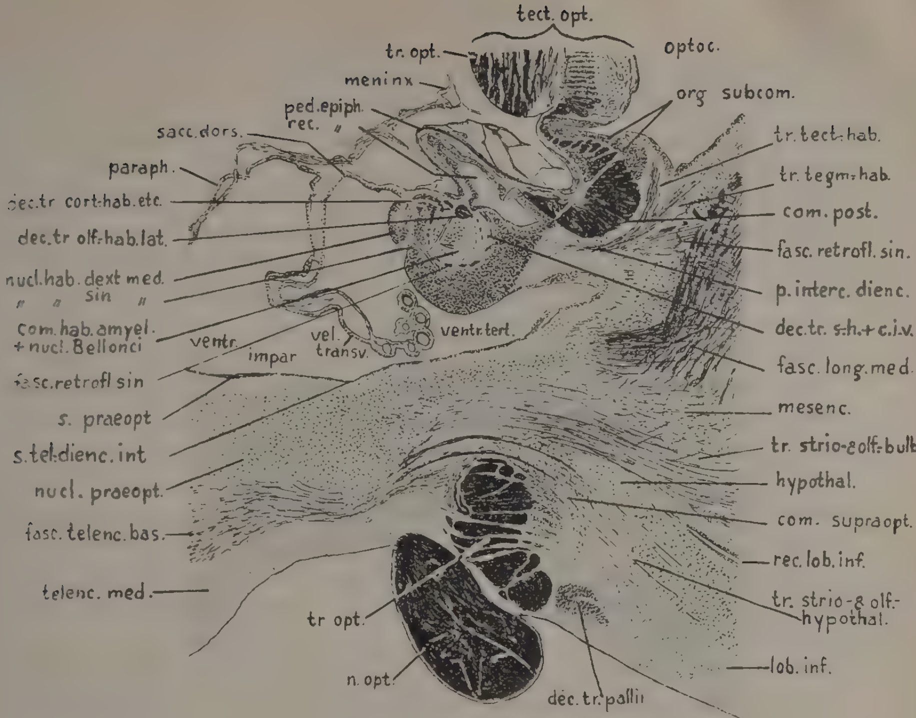
Contrary to the right bundle, the left fasciculus retroflexus chiefly is made up of unmyelinated fibres, but comparatively few myelinated ones being present in it. Thus it is considerably smaller and less conspicuous than the right bundle. As on the right side the myelinated fibres arise exclusively from the pars dorsalis of the lateral nucleus, which they also leave in the same way as on the right side. Thus the more lateral fibres of the bundle are the first to emerge, while those from the hindmost part of the nucleus, lying above the commissura habenularum, curve around in front of the latter before emerging. Like the right bundle the left at first is a broad, flat band, which caudally gradually becomes round. In the flat portion the myelinated fibres are evenly spread in small groups among the bundles of unmyelinated fibres, in the round portion they have shifted for a large part to the periphery of the fascicle, though not exclusively so.

The other efferent tract of the ganglion habenulae generally described in Vertebrates, the tractus habenulo-thalamicus, reported for Selachians by JOHNSTON (1911), could not be ascertained in my preparations.

The stria medullaris.

By far the majority of the afferent tracts of the ganglion habenulae come from before, from the telencephalon, and thus chiefly are of an olfactory nature. Already before reaching the ganglion these olfacto- and cortico-habenular tracts for the most part have united to the very large stria medullaris, which in the ganglion forms the commissura habenularum or superior.

Figs. 5 and 6. *Scyllium canicula.* Sagittal sections through ganglia habenulae and fused nuclei of BELLONCI. WEIGERT–PAL-*paracarmine*. × 21. The magnification is so chosen that the dimensions are the same as in the figures of the transverse series (figs. 1—4).
Fig. 5. Through left ganglion habenulae some sections to the left of the median plane. The rostral end of the medial right habenular nucleus, curving in front of the medial left one, a little transgresses the median plane.
Fig. 6. Through right ganglion habenulae at approximately one fifth of its width to the right of the median plane. The dorsal large-celled part of the lateral nucleus has just appeared. nucl.hab.d.m., nucleus habenularis dexter medialis; dec.tr.s.-h. + c.i.v. see explanation of abbreviations of figs. 1—4.


tect. opt.
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sacc. dors.
tr. tect.-hab.
paraph.
tr. tegm.-hab.
dec. tr cort.-hab. etc.
com. post.
dec. tr olf.-hab. lat.
fasc. retrofl. sin.
nucl. hab. dext med.
" " sin "
p. interc. dienc.
com. hab. amyel.
+ nucl. Bellonci
dec. tr. s.-h. + c. i. v.
ventr.
impar
vel.
transv.
ventr. tert.
fasc. long. med.
fasc. retrofl sin
s. praeopt
mesenc.
s. tel.-dienc. int
tr. strio-& olf.-bulb.
nucl. praeopt.
hypothal.
com. supraopt.
fasc. telenc. bas.
rec. lob. inf.
telenc. med.
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hypothal.
tr opt.
lob. inf.
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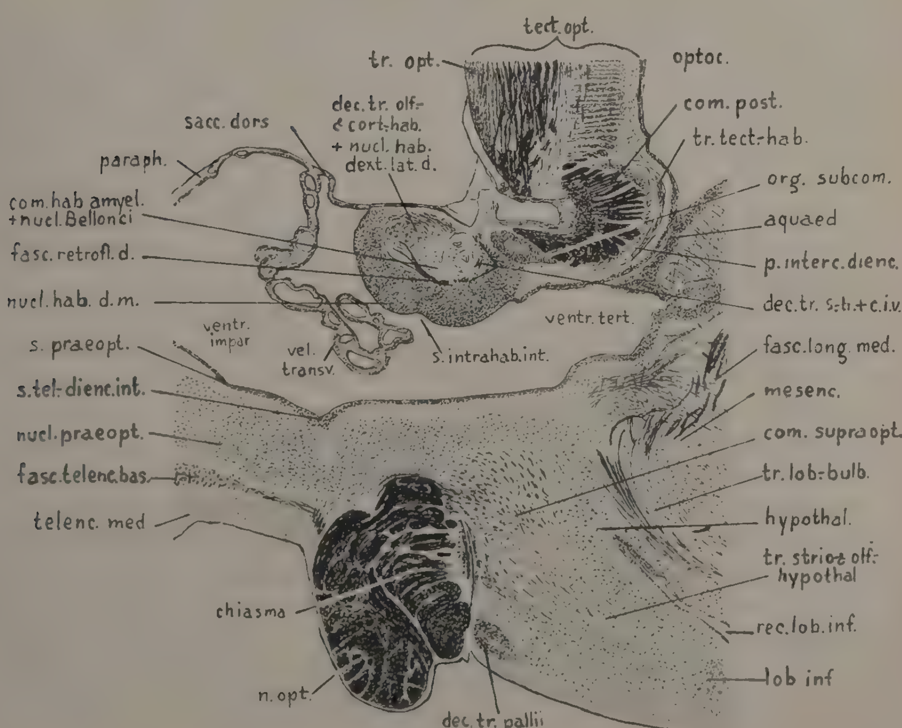
dec. tr. pallii

Fig. 5.


tect. opt.
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& cort.-hab.
+ nucl. hab.
dext. lat. d.
sacc. dors
tr. tect.-hab.
paraph.
org. subcom.
com. hab. amyel.
+ nucl. Bellonci
aquaed
fasc. retrofl. d.
p. interc. dienc.
nucl. hab. d. m.
dec. tr. s.-h. + c. i. v.
ventr. tert.
ventr.
impar
fasc. long. med.
s. praeopt.
vel.
transv.
s. intrahab. int.
mesenc.
s. tel.-dienc. int.
com. supraopt.
nucl. praeopt.
tr. lob.-bulb.
fasc. telenc. bas.
hypothal.
telenc. med
tr. strio-& olf.-
hypothal
chiasma
rec. lob. inf.
lob inf
n. opt.
dec. tr. pallii

Fig. 6.

It must be pointed out that the terms stria medullaris and commissura habenularum are used here in a broader sense than in higher Vertebrates, since, in Selachians, as in Cyclostomes, these bundles are largely composed of commissural fibres of the primordium hippocampi.

The various tracts making up the stria medullaris, principally are known by the researches of JOHNSTON (1911), who to a great extent succeeded in clearing up the composition of this complicated fibre system. In the following description, however, the more appropriate nomenclature devised by HERRICK (1910, '33) is adopted.

The stria medullaris is made up of myelinated and unmyelinated fibres, of which the latter form by far the majority.

We shall first try to analyse the myelinated part of the stria medullaris. On cross-sections through the beginning of the ganglion habenulae (fig. 1), as also on longitudinal sections through the commissura habenularum (figs. 5—7, 8), the majority of the myelinated fibres is seen covering as a cap the upper side of the unmyelinated part of the stria. On closer inspection, however, this cap appears to consist of two separate bundles, of about equal size, one dorsomedial and the other dorsolateral to the unmyelinated stria.

The dorsomedial bundle, together with the unmyelinated part of the stria, may be followed forward in horizontal direction along the dorsal margin of the telencephalon medium, and appears to arise from three components, one of dorsal, the other two of ventral origin.

The dorsal component, according to HERRICK's nomenclature, is the tractus cortico-habenularis medialis, the tractus cortico-habenularis simply of JOHNSTON. This tract arises from the primordium hippocampi, collecting in its hindmost part, immediately lateral to the ventricle. Also the unmyelinated part of the stria arises in the primordium hippocampi, chiefly at least. Like the fibres of the tractus cortico-habenularis medialis the unmyelinated fibres collect caudally in the primordium hippocampi. Already here, at their beginning, the myelinated bundle lies at the medial side of the unmyelinated.

On leaving the primordium hippocampi the comparatively weak, but well-circumscribed tractus cortico-habenularis medialis is joined by the foremost of the two ventral components of the medial myelinated bundle, viz., the tractus olfacto-habenularis of JOHNSTON. Its fibres arise from the tuberculum olfactorium and the lateral olfactory nucleus, and immediately behind the foramen interventriculare ascend to partake in the formation of the stria. This bundle is regarded by me as the combined tractus cortico-habenularis lateralis and olfacto-habenularis anterior, in HERRICK's terminology, the former being represented by the fibres from the lateral olfactory nucleus, the latter by those from the tuberculum olfactorium.

The hindmost of the two bundles reinforcing the medial myelinated

part of the stria during its course in the telencephalon medium, is called tractus taeniae by JOHNSTON and, according to him, probably corresponds to HERRICK's tractus habenulo-striaticus, but may include the tractus cortico-habenularis lateralis of this author. In my opinion it is the combined tractus strio- and amygdalo-habenularis of HERRICK, which are joined by a part of the tractus olfacto-habenularis medialis, coming from behind. The latter arises from the nucleus praeopticus and runs medial to the basal forebrain bundle.

Contrary to the medial myelinated bundle, the lateral bundle is of single origin. It is the tractus olfacto-habenularis lateralis of HERRICK (posterior of JOHNSTON), which, like the tractus olfacto-habenularis medialis, arises from the nucleus praeopticus, but runs lateral to the basal forebrain bundle. The fibres composing it ascend parallel to the periphery of the telencephalon medium, running partly through and partly medial to the tractus pallii. More caudally they also come forth between the bundles of the optic tract. [1])

In the transverse WEIGERT-PAL-paracarmine series of Scyllium a small fibre bundle was observed seeming to come from the optic tract and joining the tractus-habenularis lateralis, which in its compactness quite resembled a bundle of the optic tract. On closer inspection, however, it appeared merely to be a small bundle of the tractus olfacto-habenularis lateralis, ascending through the tractus pallii.

Having thus ascertained the origin and nature of the two principal myelinated fibre bundles of the stria, we now shall describe their behaviour in the commissura habenularum. Here they form a very asymmetrical decussation fronto-dorsal to the unmyelinated stria, the medial bundle crossing foremost (figs. 3, 5—8).

On the right side the two bundles, having entered the pars dorsalis (the large-celled part) of the lateral nucleus, indistinguishibly fuse, their fibres diverging and splitting up, and forming with the emerging myelinated fibres of the fasciculus retroflexus the network, by which this part of the nucleus stands out so strikingly in myelin stained preparations (figs. 1—3, 6). Having passed the median plane the two bundles resume their compactness, and again become independent of one another (fig. 3). Evidently most of their fibres, not only of the right but also of the left bundles, end in the large-celled right nucleus, synapsing there with the neurons of the myelinated part of the fasciculus retroflexus. In Petromyzonts, according to JOHNSTON (1913), the left stria, sensu strictiori, of course (i. e. without the commissural fibres of the primordium hippocampi), almost exclusively goes to the right ganglion.

Besides the medial and lateral bundles just described, which constitute

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[1]) In addition to the principal bundle there is a separate posterior part of the tractus olfacto-habenularis lateralis, which will be discussed in the second part of the paper in the section on the nucleus commissurae hippocampi.

the majority of the myelinated fibres of the stria medullaris, there are yet other myelinated fibres partaking in the formation of this bundle, or better said of the commissura habenularum. They cross in its hindmost part (figs. 3—8) and belong to two categories, those lying at the periphery of the commissure being of a different nature from those lying more inward.

We shall first deal with the peripheral fibres. They are united to small bundles, which run through or immediately over the fasciculus retroflexus, thus making the impression of being crossing fibres of the latter (figs. 3, 4). The hindmost of them, in cross-sections, may be confused with the foremost fibres of the commissura posterior, which begins to appear at this level. The fibres in question, however, run ventral to the recessus epiphyseos, between this and the ganglion habenulae (figs. 4, 5—8), whereas the foremost fibres of the commissura posterior run dorsal to the subcommissural organ, into which the recessus epiphyseos opens (figs. 5—8).

A small part of the foremost of these peripheral myelinated fibres of the stria, having arrived at the median plane, ascend on either side of it, and at the level of the dorsal lateral habenular nuclei go to the other side and end in the latter.

The ultimate origin of these fibres could not be determined. They come from before, ventromedial to the unmyelinated stria (figs. 1, 2), and, in our preparations, a little in front of the ganglion habenulae are lost out of sight. No doubt they are identical with the fibre bundle JOHNSTON describes as leaving the medial forebrain bundle just in front of the ganglion habenulae. JOHNSTON per exclusionem considers them as tractus septo-habenularis, an interpretation which seems rather plausible and, with some reserve, is adopted by us [1]).

Figs. 7 and 8. *Scyllium canicula.* Projections on the median plane of left (fig. 7) and right (fig. 8) ganglion habenulae and their environment. × 22. For these reconstructions the transverse WEIGERT–PAL-paracarmine series was used after which figs. 1—4 were drawn, whose position is indicated by interrupted lines. Since in this series no indicator lines were traced, the reconstructions were made with the aid of the sagittal WEIGERT–PAL-paracarmine series. As in Necturus (1943, fig. 8) and Bdellostoma (1943a, fig. 2) the magnification was so chosen that the distance between the front of the nucleus commissurae hippocampi (in 1943 called eminentia thalami) and the end of the commissura posterior is equal to the average of these distances on the left and right side in Petromyzon fluviatilis (1943, figs. 6, 7), which are approximately the same. d.c.-h., decussation of lateral and medial cortico-habenular, medial and anterior olfacto-habenular, and amygdalo- and strio-habenular tracts; d.o.-h., decussation of lateral olfacto-habenular tract; dec.tr.sept.-hab. + c.i.v., see explanation of abbreviations of figs. 1—4.

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[1]) In JOHNSTON's fig. 10, by a lapse, the designations of the tractus septo-habenularis and the crescent-shaped fibre bundle he calls optic radiation, are interchanged.

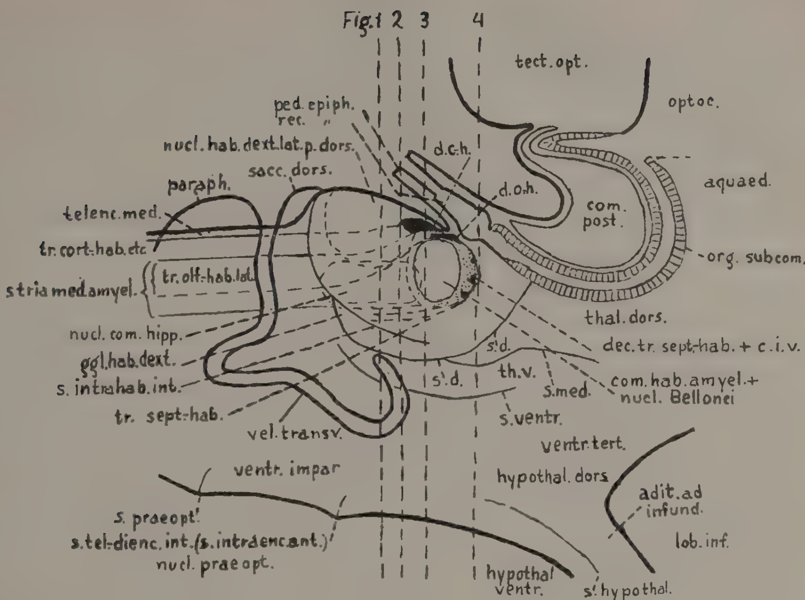
Fig. 1 2 3 4
tect. opt.
optoc.
ped. epiph.
rec.
nucl. hab. dext. lat. p. dors.
d.c.h.
sacc. dors.
aquaed.
paraph.
d.o.h.
com. post.
telenc. med.
tr. cort.-hab. etc.
org. subcom.
tr. olf.-hab. lat.
stria med. amyel.
thal. dors.
nucl. com. hipp.
dec. tr. sept.-hab. + c.i.v.
ggl. hab. dext.
s. intrahab. int.
th. v.
s. med.
com. hab. amyel. +
nucl. Bellonei
tr. sept.-hab.
s. ventr.
vel. transv.
ventr. tert.
ventr. impar
hypothal. dors.
adit. ad infund.
s. praeopt.
s. tel.-dienc. int. (s. intraenc. ant.)
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nucl. praeopt.
hypothal ventr.
s. hypothal.

Fig. 7.

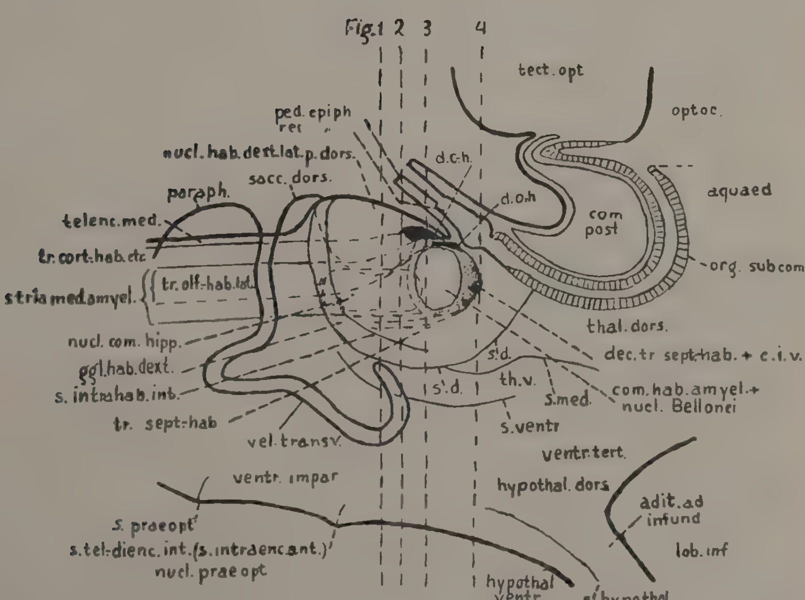
Fig. 1 2 3 4
tect. opt
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com post
telenc. med.
tr. cort.-hab. etc.
org. subcom.
tr. olf.-hab. lat.
stria med. amyel.
thal. dors.
nucl. com. hipp.
dec. tr sept.-hab. + c.i.v.
ggl. hab. dext.
s. intrahab. int.
th. v.
s. med.
com. hab. amyel. +
nucl. Bellonei
tr. sept.-hab
s. ventr
vel. transv.
ventr. tert.
ventr. impar
hypothal. dors
adit. ad infund
s. praeopt.
s. tel.-dienc. int. (s. intraenc. ant.)
lob. inf
nucl. praeopt
hypothal ventr.
s. hypothal.

Fig. 8.

Also BÄCKSTRÖM (1924) may have seen these fibres, which be describes as a collection of fine fascicles given off by the medial forebrain bundle to the stria medullaris. He, however, interprets them as tractus cortico-habenularis, meaning with this term the tractus cortico-habenularis medialis of HERRICK's terminology, the bundle, consequently, arising from the primordium hippocampi. The tract described by JOHNSTON as tractus cortico-habenularis (medialis HERRICK) is called by BÄCKSTRÖM tractus pallialis praeopticus. As clear from the above, we share JOHNSTON's interpretation of the two bundles concerned, though with some reserve in the case of the tractus septo-habenularis.

The second category of myelinated fibres crossing in the hindmost part of the commissura habenularum, viz., those lying more inward, consists of widely scattered single fibres (figs. 3, 5—8). They come forth medial to and from the inner layers of the optic tract (fig. 3), and are held by me to be derived from the tractus olfacto-habenularis medialis. When dealing with the nucleus of BELLONCI, I shall return to them at large.

We now proceed to the unmyelinated part of the stria medullaris, which forms by far the larger part of the bundle. Accompanied by the lateral and medial myelinated bundles it comes from before along the dorsal margin of the telencephalon medium.

The unmyelinated stria generally is held to consist of commissural fibres only. According to JOHNSTON (1911), followed in this by STUART THOMSON (1919), it is a commissure of the primordium hippocampi, but according to BÄCKSTRÖM (1924) it contains, in addition, a general pallium commissure. These authors call it commissura pallii posterior. KAPPERS, on the other hand, holds that the unmyelinated stria represents the commissural fibres of the primordium piriforme, which he calls commissura superior telencephali.

My attitude in this matter is conciliatory. No doubt the unmyelinated part of the stria largely is made up of commissural hippocampal fibres, but I think it quite probable that the bundle is reinforced by ventral fibres, representing the commissura superior telencephali of KAPPERS. The presence, on the contrary, of a general pallium commissure, as maintained by BÄCKSTRÖM, I think excluded. In addition, however, to the commissural fibres, in my opinion, unmyelinated fibres, either direct or decussating, of the various habenulo-petal tracts may partake in the formation of this large bundle.

Closing the description of the components of the stria medullaris, a remark has yet to be made about the position of its myelinated and unmyelinated fibres with regard to each other. According to KAPPERS, in the commissura habenularum the myelinated fibres surround the unmyelinated bundle like at thick myelin sheath surrounds a big axis-cylinder, a behaviour, according to this author, often met with when myelinated and unmyelinated fibres run together. In the case under

consideration this statement holds good for the majority of fibres, our medial and lateral myelinated bundles and also the hindmost part of the posterior crossing fibres (the tractus septo-habenularis) lying at the periphery of the unmyelinated bundle. The fibres, however, derived from the tractus olfacto-habenularis medialis, lie inside it, as also is plain from KAPPERS' figure (1920, fig. 41). Moreover, the myelinated fibres do not form a closed ring, as indicated in this figure, the frontal und ventral ones of them being emerging fibres of the fasciculus retroflexus (figs. 5, 6).

The tractus tecto-habenularis.

In addition to its principal impulses, chiefly olfactory, coming from before by way of the telencephalic habenulo-petal tracts united to the stria medullaris, the ganglion habenulae also receives impulses from behind, from the mesencephalon. They are carried by a bundle of loosely arranged myelinated fibres, which are derived from two sources. The majority arise from the tectum opticum and represent the tractus tecto-habenularis. They run, on either side of the subcommissural organ, through or immediately ventral to the pars intercalaris diencephali and thus, like the latter, the commissura posterior and the subcommissural organ, are horse-shoe-shaped (figs. 5, 6).

The bundle described by us as tractus tecto-habenularis, was already seen by EDINGER (1892), who, however, regarded it as running in the opposite direction and accordingly called it tractus ganglii habenulae ad mesocephalon dorsalis. The large fibre bundle, described by B. HALLER (1898) in Scyllium canicula under the latter name, which according to him, can be traced into the region of the oculomotor, only for a small part corresponds to our tract, being almost entirely an optic bundle, running on the medial side of the tectum opticum.

The tractus tegmento-habenularis.

The second group of fibres going from the mesencephalon to the ganglion habenulae, comes from the tegmentum and joins the tractus tecto-habenularis at its convexity (fig. 5). As far as I know such a tract hitherto has not been described. KAPPERS, HUBER and CROSBY (1936), however, in the first volume of their text-book, without further reference, mention a tractus habenulo-tegmentalis, going to the dorsal tegmental nucleus of VON GUDDEN.

Anatomy. — *The model of* LILLIE *in connection with the growth of the nerve-fibre.* By R. BRUMMELKAMP. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of October 27, 1945.)

I.

LILLIE has invented a model by the help of which numerous nervous phenomena can be imitated. In its most simple form this model consists of a stretched iron wire (piano-string), which is plunged into nitric acid of a certain strength (specific gravity 1.42), so that on its surface a thin layer of high iron-oxide is formed. If one "excites" such a wire by damaging this oxide-layer locally by an electrical, chemical, thermical or mechanical impulse, a passing oxidation-reduction-process runs across its surface, which, physically speaking, shows a striking resemblance with the "condition of excitation" proceeding along an excited nerve-fibre. The correspondence is so great that one is inclined to consider the physical powers which work in both processes as identical.

The rapidity with which both processes proceed is of the same order; the all-or-none-law obtains for he excited place; each exited place returns automatically into the condition of rest; during this return there is a total or partly refractional period; at all points similar reactions follow one another, accompanied by an electric phenomenon; the reaction is introduced by a change in the permeability of a surface layer; the excitatory current can be delayed, resp. blocked by an electric current, also by a chemical action; during a partly refractional period currents with a decrement can arise; a certain minimal intensity is necessary for a local excitation; without a local excitation no proceeding current arises; several stimuli in themselves subliminar and following one another become active by summation; an electric current of too little strength, independent of the duration of its action, has no effect; it is necessary that a supraliminary current acts for some time; among other things the duration of the action depends on the strength of the current used; the model shows the phenomenon of the gradually increasing impulse; a place which is in a condition of excitation shows a negative charge with respect to the non-excited part. All these properties are characteristic both of the nerve-fibre and of LILLIE's model.

I have asked myself whether perhaps, besides the physiological properties of the nerve-fibre, the way of growing of the nerve-fibre can also be imitated in the model.

SPEIDEL has studied the growth of the nerve-fibre daily by observing

the tail-fin of the larva of a frog. He found a) that the fibre at its extremity grows out by lengthening itself (in the direction of the axis of the fibre); b) that the side-branches which spring from the fibre form an angle with the axis of the fibre, approaching more and more a right angle as the side-branches spring further from the extremity of the fibre, so that c) at some disctance of the extremity the side-branches stand perpendicularly on the axis of the fibre (fig. 1). The movements which the free

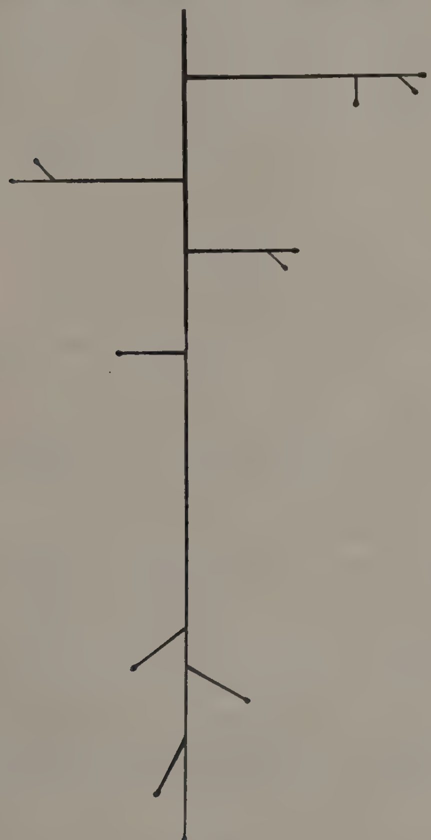

Fig. 1.
Schematic growth of a nerve fibre.

extremities of the fibre and the side-branches make in their growth give the impression as if the way is sought gropingly, while small disturbances must be conquered and the extremities must turn aside now to the right and then again to the left over small distances. One gets the impression as if there are invisible (submicroscopic) structures which influence the way which the nerve takes at a given moment.

Bok showed already before that the perpendicular growth mentioned under c) is also found in the neurites of young neuroblasts, situated in the neighbourhood of a fibre-bundle.

Weiss proved by experimentation that the direction of the movements

of the offschoots of a young neuroblast can be determined by the microstructure of the surroundings. The experiments of WEISS are in agreement with the observations of HARRISON; both say that the direction of the growth of a nerve-fibre orientates itself to the structures present in the surroundings; with WEISS these are directed micells, with HARRISON fine cobweb-threads.

It will be explained below that by means of the iron wire of LILLIE field-powers arise which act in a directing manner on little bipolar bars, present in the surroundings of the wire and the question is asked whether the nerve-fibre may also be supposed to have such directing powers, which may then lead to such changes in the microstructure of the surroundings of the nerve-fibre, resp. of the positions of the bipolar micells. present in the nerve-fibre itself and that from this the way of growing of the neurite can be derived.

Experimental part.

The following experiments were made.

A shallow glass bowl of 20 by 15 by 5 cm. is half filled with nitric acid and into this a bundle of about ten piano-strings, each 1 mm. thick and 14 cm. long, is plunged. There arises a thin, metal-like layer of a high iron-oxide on the surface of the wires.

We now bring a small iron bar of 3 cm. long in the surrounding electrolyte, after one half of the bar has been protected from the electrolyte. by enveloping this half with a layer of paraffin. This iron bar is kept floating by mounting it on a paraffined piece of cork. Under the influence of the electrolyte the non-paraffined part of the bar will also envelop itself with a layer of high iron-oxide and thus obtain a surface-charge equal to that of the membrane of the iron bundle, while the paraffined (the non-oxidated) part takes an opposite charge. So the bar has the function of a small dipole. (A non-paraffined bar can also act as a bipole, when the charge accumulates at one pointed extremity.)

In these experiments the level of the nitric acid is so high that the floating bar and the iron bundle lying on the bottom of the bowl do not touch.

If we place the bar opposite the middle of the bundle and 5 cm. laterally from this, parallel to the bundle, the bar turns 90 degrees until its paraffined end is directed towards the bundle and then moves towards the bundle, after which it remains standing perpendicularly on the bundle (fig. 2). The intensity of this attraction appears among other things to be dependent on the number of wires of which the bundle consists.

If we place the bar with its non-paraffined part opposite the middle of the bundle, perpendicularly on it, it first moves from the bundle over a distance of about 5 cm., then turns 180 degrees and then moves with its paraffined part in front, towards the bundle, after which it remains standing perpendicularly on it (fig. 3).

If we place the bar with its paraffined end perpendicularly on the bundle at a distance of 7 cm. from the free extremity (i. e. opposite the middle of the bundle) it remains standing in this position. If the distance is lessened to 6 cm., the position of the bar changes from prependicular to an angle of about 65 degrees in the direction of the extremity of the bundle. If this distance decreases to 5 cm. the angle in question becomes about 45 degrees (fig. 4).

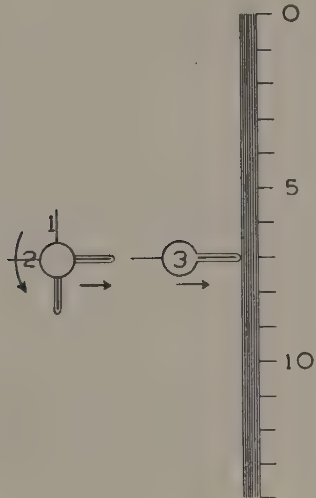

Fig. 2.
Placing of the "micells" perpendicular to the middle of the bundle.

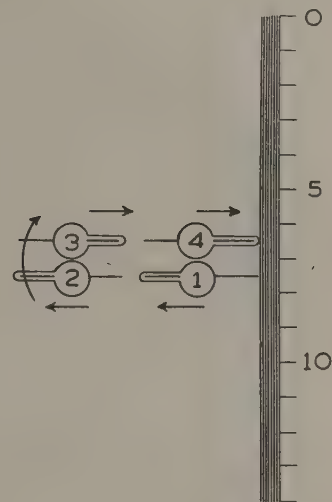

Fig. 3.
Position of the "micells" perpendicular to the middle of the bundle.

If we now place the bar a little more in the direction of the extremity, it shifts forward spontaneously in its entirety and puts itself in the longitudinal axis of the bundle; then it moves at first slowly, but gradually quicker, towards the extremity of the bundle where it stops as soon as the middle of the bar has nearly reached the free extremity of the bundle (fig. 5).

If we put the same bar alternately at the two extremities of the bundle, it takes up the positions of fig. 6. If the phenomenon was of a magnetic kind, the bar would be have to take up the same position at both extremities of the bundle; if, on the contrary, electrostatic powers work, the positions are equal to those of fig. 6. The positions of the bar evidently correspond to the power-lines of the electric field round the iron bundle.

If two bars are in the bowl, they either put themselves in such a position that one is the lengthening-piece of the other, with the paraffined end of the one bar against the non-paraffined end of the other, or they put themselves beside each other, so that the opposite poles touch. If there are more bars, they may form bundles, e. g. because the bars put

themselves in such à position that the one is the lengthening-piece of the other or arrange themselves in such a way beside one another that the pole of one bar comes to lie against the middle of the neighbouring bar. Thus long-drawn bundles may arise.

If there is also a bundle of wires in the bowl, the bundle of bars is inclined to stand perpendicularly on the bundle of wires, when the bars

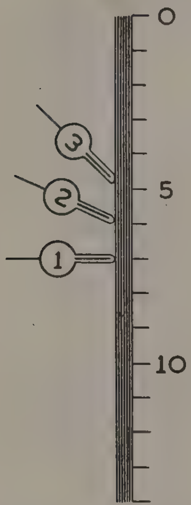

Fig. 4.
Position of the "micells" according to the field powers of the electric field.

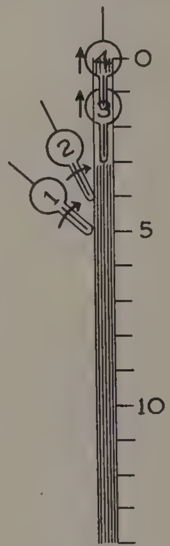

Fig. 5.
Movement of the "micells" to the free extremity of the bundle.

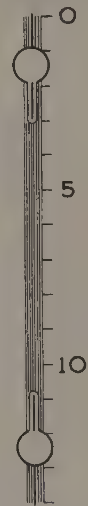

Fig. 6.
Position of the "micells" from which is concluded that the phenomenon has to be ascribed to electric and not to magnetic forces.

are opposite the middle of the bundle of wires; if the distance towards the extremity is lessened, the bundle of bars nestles up to the bundle of wires and then replaces itself in its entirety some distance towards the extremity of the bundle of wires (fig. 7 and 8). If in this case the bundle of bars is so long that one of its extremities reaches across the middle of the bundle of wires, this end is swung laterally, until it stands perpendicularly on the bundle of wires (fig. 9).

If we suspend a bar in the bowl on a glass wire, in the neighbourhood of the bundle, this bar is repulsed. If we now damage the surface-membrane at one of the extremities so that an oxidation-reduction-process runs across the bundle (current of negativity) the bar is attracted by the wire as soon as the current of negativity reaches the bar. This attraction increases in proportion as the bundle contains more wires.

Summarising we can say that the bars under the influence of the surface-charge of the iron bundle move towards it, placing themselves in accordance with the power-lines of the electric field round the bundle.

i. e. at the extremity of the bundle as its lengthening-piece and laterally from the bundle under angles which approach more to 90 degrees in proportion as the distance from the extremity of the bundle increases. Equal charge of bar and bundle causes repulsion, opposite charge attraction. Excitation of the bundle leads to attraction of a bar which was at one time repulsed, as soon as the condition of excitation runs along the bar.

It appears among other things from the above-mentioned experiments that the positions which the bipolar bars take up, correspond with the positions which the growing neurite shows at its extremity and at its side-branches.

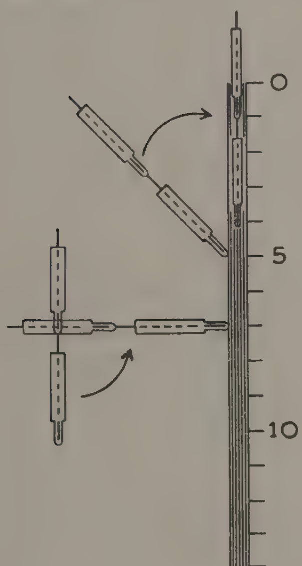

Fig. 7.
Placing of a pair of "micells".

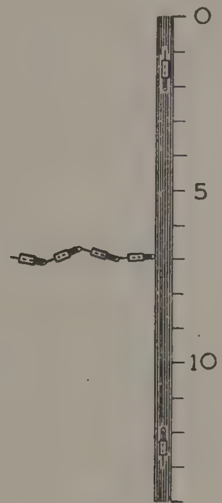

Fig. 8.
Position of four "micells" perpendicular to the middle of the bundle.

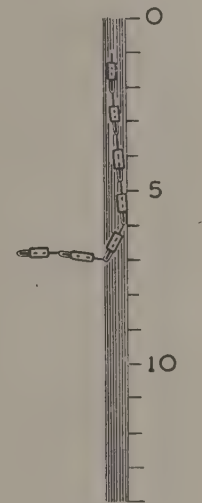

Fig. 9.
Thickening of the bundle by apposition of "micells" and the arising of an offshoot in the midst of it.

If we suppose that the nerve-bundle in vivo carries a surface-charge and that in the surroundings of the nerve-bundle there are bipolar micells in the form of bars (macromolecular albumina) an arrangement [1]) of these micells must arise corresponding to that of the bars of our model-experiments, an arrangement which, as we said already, corresponds with the direction of the growth of the neurite and its side-branches. We can now easily understand (see experiment of WEISS) that led by the microstructures of the surroundings the nerve-fibres seek their way through the surroundings, giving expression to the arrangement of the macromolecules of the surrounding medium.

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[1]) This arrangement could perhaps be experienced by means of the polarisation-microscope.

It is clear, however, that the field of action of the surface-charge is not limited to the micro-structures of the surroundings, but also exerts a directing influence on the micells, occuring in the offshoots of the nerve-fibre itself [2]. *This directing influence generally coincides with that which is exerted on the micells of the surroundings, so that the two actions strengthen each other.*

II.

To the above we add the following supplementary remark. It has appeared from experiments made by BURR, DETWILER, JULIUS a. o. that there is an attracting action between nerve-fibres and quickly growing cells (normotaxis); this attracting, action is strengthened if these are nerve-cells, which together with the nerve-fibre are in a condition of excitation (*neurobiotaxis*, ARIËNS KAPPERS). Probably the normotaxis should be attribued, at least part, to the high metabolsm of the quickly growing cells. Here acids form themselves, which, by driving the H'-ions back, give a positive charge to the cellular colloids. In the nerve-fibre this condition is not fulfilled; thus an electric potential-difference arises between the cell and the fibre. The *neurobiotaxis* can be considered as a strengthened form of normotaxis, for if a nerve-fibre and a neighbouring nerve-cell are continuously excited, currents of negativity continuously run across the nerve-surface and at the same time the cellular metabolism is heightened (greater formation of acids, higher positive charge of the cellular colloids), so that the potential-difference between cell and fibre increases. From our experiment with the iron bundle of LILLIE it appears that the potential-difference between the "wire that is being excited" and an iron bar suspended opposite this wire is sufficient to bring about an attracting action over a macroscopic distance. Mutatis mutandis the same can certainly be the case when the microscopic distance between an excited nerve-fibre and a neighbouring nerve-cell is bridged. Unfortunately quantitative data concerning this are still lacking.

Summary.

LILLIE's model does not only give a clue for explaining the physiological properties of nerve conduction but may also help us to explain the way of growing of the nerve fibre.

The simplest way is to consider the powers which influence the growth of the nerve fibre as powers in an electric field.

The micro-structure of the surroundings of the nerve fibre can be derived from these field powers. The same obtains for the arrangement of the micells partaking in the structure of the offshoots of the nerve

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[2]) In this aspect we refer to the experiments of HALLIBURTON who found that anhydrating of the neuron axis leads to a regular arrangement of the enclosed micells from which it is concluded that these micells have the shape of a bar (baccolites).

fibres. LILLIE's model and the nerve fibre correspond in having the same spatial pattern.

As the experiments described with the lifeless model only allow of a conclusion made by analogy and are not as valuable as experiments on the living object, the correspondence between the observations recorded in this paper and the growth of the nerve is striking.

The phenomenon of neurobiotaxis finds a place fitting in with the whole of our considerations.

Samenvatting.

Het model van LILLIE opent niet slechts de mogelijkheid een verklaring te geven van de physiologische eigenschappen van een zenuwvezel, doch kan ons ook een inzicht verschaffen in de wijze van groei van den zenuwvezel.

Het eenvoudigst komt men uit door de krachten, die invloed hebben op den groei van den zenuwvezel, op te vatten als krachten in een electrisch veld.

De microstructuur van het milieu rondom den zenuwvezel kan uit deze veldkrachten worden afgeleid. Dit zelfde geldt voor de rangschikking van de micellen, waaruit de uitspruitsels der zenuwvezels zijn opgebouwd. Beide structuren hebben eenzelfde ruimtelijk patroon en dekken elkaar.

Ofschoon de beschreven proeven met het levenlooze model natuurlijk slechts een analogie-besluit toelaten en niet de waarde hebben van proeven op het levende object, is toch de overeenkomst tusschen de waarnemingen, waarover hier bericht werd, en den groei van den zenuwvezel opvallend goed.

Het verschijnsel der neurobiotaxis vindt in het kader onzer beschouwingen een plaats.

Zusammenfassung.

Das Modell von LILLIE erschlieszt nicht nur die Möglichkeit eine Erklärung der physiologischen Eigenschaften einer Nervenfaser zu geben, sondern kann uns auch einen Einblick in das Wachstum der Nervenfaser vermitteln.

Am zweckmäszigsten kommt man dadurch dazu, dasz man die Kräfte, die auf das Wachstum der Nervenfaser Einflusz haben, als Kräfte in einem elektrischen Feld auffaszt.

Die Mikrostruktur der Umgebung der Nervenfaser kann aus diesen Feldkräften abgeleitet werden. Dasselbe gilt für die Anordnung der Mizellen, aus denen sich die Ausläufer der Nervenfasern zusammensetzen. Beide Strukturen haben ein gleiches Raummuster und decken sich.

Obwohl die beschriebenen Experimente am lebenlosen Modell natürlich blosz einen Analogieschlusz erlauben und nicht den Wert der Experimente am lebenden Objekt haben, ist dennoch die Übereinstimmung.

zwischen den Beobachtungen worüber hier berichtet wird und dem Wachstum der Nervenfaser, auffallend gut.

Das Phänomen der Neurobiotaxis findet im Rahmen unserer Betrachtungen einen Platz.

Résumé.

Le modèle de LILLIE ouvre non seulement la possibilité de donner une explication des propriétés physiologiques du fibre nerveux, mais encore il peut nous faire comprendre la façon de croître de ce fibre.

Le plus simple pour nous est de se représenter les forces qui influent la croissance du fibre nerveux comme des forces dans un champ électrique.

La microstructure du milieu entourant le fibre nerveux peut être déduit de ces forces. Ceci est de même pour l'arrangement des micelles qui constituent les bourgeons des fibres nerveux. Les deux structures ont le même ordre et se couvrent.

Quoique les expériences avec le modèle, que nous avons décrites, ne permettent naturellement qu'une conclusion analogique et n'aient pas la valeur d'expériences sur des objets vivants, quand-même l'analogie entre les observations mentionnés ci-dessus et la croissance du fibre nerveux est remarquablement bonne.

Le phénomène du neurobiotaxis trouve une place dans le cadre de nos considérations.

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Anatomy. — *Comparative anatomy of the caudal part of the human and subprimate cerebellum.* By J. M. SCHOLTEN. (Communicated by Prof. C. U. ARIENS KAPPERS.)

(Communicated at the meeting of October 27, 1945.)

In a former communication we tried to homologize the various portions of the cerebellum of the subprimate with those of the human cerebellum. Among other things we came to the conclusion that BRADLEY's *paraflocculus* in man is represented by the lobus biventer and the tonsil. In order to test this result still further we now shall see in how far the two lobes exhibit any similarity in their development, for, if the paraflocculus is actually homologous with the lobus biventer and the tonsil, the ontogeny of these portions of the cerebellum will undoubtedly be analogous, the cerebellum of all mammals constructed according to the same rules (under the influence of vestibular, spino-cereballar and primitive olivary fibres, to which are added impulses from higher levels reaching it via pons and principal olive).

Besides a study of the order in which the first cerebellar fissures originate has tought us that this order is the same in homo as in the subprimates, namely as follows: fissura uvolo-nodularis with her lateral elongation (fissura flocculi sive postero-lateralis), fissura prima, fissura secunda, fissura praeculminata, followed immediately by the sulcus praepyramidalis. (BRADLEY '03, '04, '05, ELLIOT SMITH '03, BOLK '06, LANGELAAN '19, JACOB '28, LARSELL & DOW '35).

Considering these two points it is highly probable that there must be also a similarity in the mode of origin between the fissures limiting the paraflocculus on one hand and the lobus biventer and tonsil on the other.

Our study on the origin of the paraflocculus in sub-primates begins at the stage when the fissura uvulo-nodularis, f. postero-lateralis and f. prima have been (are) already formed.

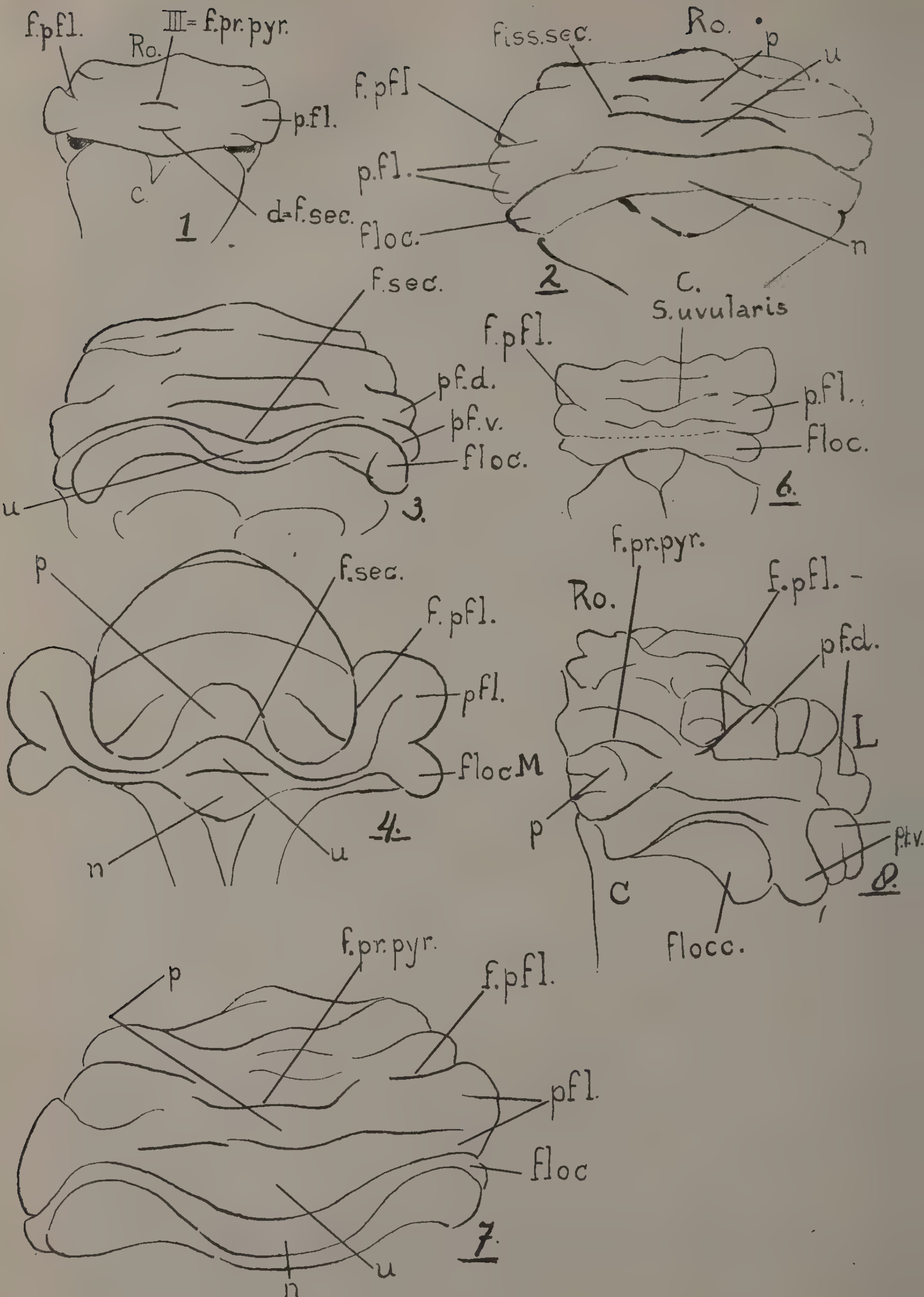
I. *The fissura parafloccularis.*

According to ELLIOT SMITH ('03) the sulcus parapyramidalis originates bilaterally. It is probable, however, that this is not the fissura parafloccularis. For this see figure 5, from which it will be evident that this fissure lies much more caudally than the others.

BRADLEY ('03, '04, '05) showed that the fissure originates bilaterally in the corpus cerebelli as this author clearly demonstrated in the cerebellum of the pig, seen from the caudal side, reproduced in our figure I (Plate I).

In the course of its further development this fissure extends medially

PLAAT I.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independant of any groove of the vermis. The same is seen in the rabbit, the sheep and the calf. Earlier STROUD ('95) showed this for the cat. LARSELL and DOW ('35) could confirm this for the opposum.

Our observations agree with those of the last mentioned authors. Fig. 2 is the cerebellum of the calf, 8½ cm. in length. The figure speaks for itself.

II. *Sulcus intratonsillaris.*

This fissure divides the paraflocculus in two parts, a ventral and a dorsal one. Fig. 2 shows that it originates laterally. Adjacent to it, the fissura secunda may be observed extending sidewards. Later these two fissures meet each other as is shown in figure 3. On the right the fissures have approached each other very nearly. Our observations are in entire agreement with those of BRADLEY. Although LARSELL and DOW make no mention of this fact, it can be deduced from their figures 15 and 16 that they, too, observed that the fissura intratonsillaris originates in this way.

III. *Fissura secunda.*

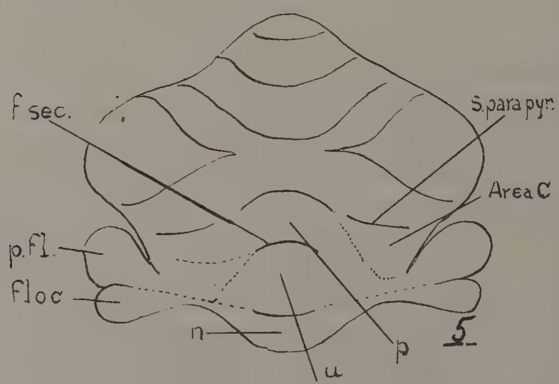
Concerning the origin of this fissure, ELLIOT SMITH ('03) has expressed various opinions. According to his first scheme — fig. 4 — this fissure is continuous with the sulcus intratonsillaris. Later, however, it appeared to him (see his second scheme — see fig. 5 left) that the fissure is continuous with the fissura floccularis — which borders the lobus flocculonodularis of the corpus cerebelli — or with a groove in the paravermis. LARSELL and DOW say that the fissura secunda extends but little laterally. BRADLEY observed that the fissure extends laterally into the paraflocculus which has meanwhile appeared, and thus — together with the sulcus intratonsillaris — divides the paraflocculus into a ventral and a dorsal part. This likewise we can confirm (see figure 3).

IV. *Fissura praepyramidalis.*

This groove also originates in the median line and develops in lateral direction. On this point the opinion of all authors agree, although not upon the question as to whether or not the sulcus praepyramidalis meets the fissura parafloccularis.

BRADLEY and ABBIE ('34) mention the confluence of the two fissures respectively in the pig and in Echidna. LARSELL and DOW hold the same opinion. According to ELLIOT SMITH the praepyramidal fissure may behave in various ways, being sometimes continuous with the fissura parafloccularis, sometimes passing through area C, — in which case it is continuous with the sulcus parapyramidalis. In the larger mammals, however, the fissura parafloccularis is not continuous with any important groove. Fig. 5 right represents both cases. Although STROUD does not mention the fact, in the figures of this author there is no continuity between the fissura para-

floccularis and his sulcus uvularis. In our opinion, however, the latter is the fissura praepyramidalis [1]) as illustrated in figures 7 and 8, which show that the fissura parafloccularis ends at some distance from the sulcus pyramidalis. The former groove if prolonged medially, will intersect the pyramis, with the result that besides the paraflocculus, another part of the corpus cerebelli coincides with the pyramis. This part is the lobus paramedianus, as the study of the adult cerebellum already taught us ('43). The connection of the lobe with the pyramis is made by a lamella hidden in the depth of the fissure between the lobus medius medianus INGVAR and the pyramis, but continuing into this last sub-lobe. At first glance it would seem as if the sulcus praepyramidalis passes over into the sulcus parafloccularis. A closer inspection, however, reveals that the sulcus proper, in the depth between the pyramis and the portion of the vermis lying rostrally to it, is bridged laterally by the aforesaid lamella



of the lobus paramedianus. The merging of the two grooves is thus merely a seeming one. This, in our opinion, is the explanation of the different statements which are to be found in literature [2]).

V. The development of the paraflocculus of the subprimates takes place as follows: Shortly after the fissura secunda appears in the median line, the praepyramidal sulcus can be seen, also first in the median plane. Soon after the sulcus parafloccularis makes its appearance, bilaterally, in contrast with the two former. In this way a lobe is cut off from the

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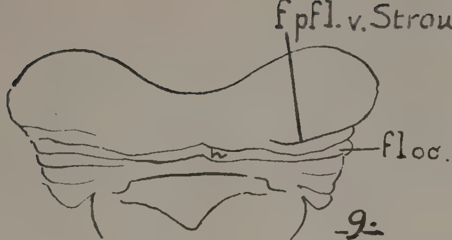
[1]) The term sulcus uvularis for this fissure is used by *Stroud* because it bounds the uvula rostrally. The pyramis according to *Stroud* is rudimentary in the animal in question, the cat. This, however, is not the case as can be seen from the arbor vitae. Besides the sulcus uvularis exhibits the typical appearance of the sulcus praepyramidalis for which reason we consider it to represent this sulcus.

[2]) For another point of importance, our conception of a secundary fissure, see sub X.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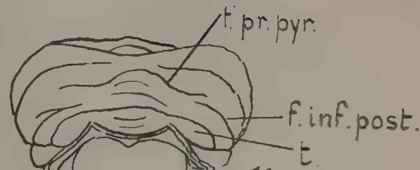
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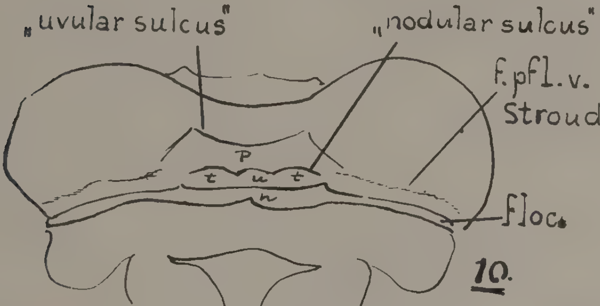
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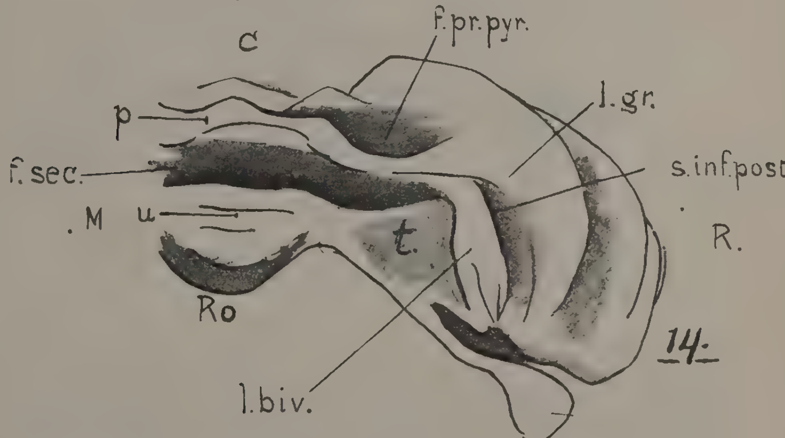
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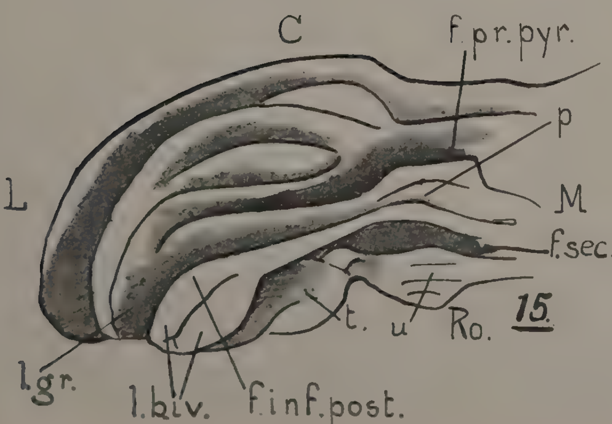
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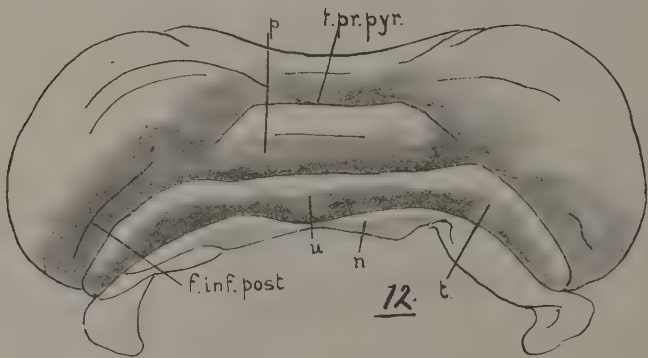
corpus cerebelli. This lobe, the paraflocculus, is bounded caudally by the fissura postero-lateralis (s. flocculаris) already present at that time.

The fissura parafloccularis extends itself medially, thus demarcating more and more the paraflocculus. At the same time the fissura secunda extends further laterally as far as into this lobe thus dividing it into two parts. Thereby this fissure becomes continuous with the sulcus intratonsillaris, which had already begun to form in the paraflocculus.

Like the fissura secunda also the sulcus praepyramidalis grows laterally. It terminates at some distance from the median line and this does not meet the fissura parafloccularis. This groove — if prolonged sufficiently — will intersect the pyramis.

We shall now turn our attention to the human cerebellum commencing with a consideration of the



VI. *Sulcus inferior posterior,*

separating the lobus biventer from the rest of the hemispheres.

Bradley found the origin of this fissure to be bilateral as is clearly demonstrated in one of his figures (see fig. 11). From the figures given by Langelaan ('19) this author observed the same, as did Ingvar ('18). According to Bolk ('06) this groove is formed by the sulcus praepyramidalis which extends laterally, which view is shared by Jacob ('28). We, however, were able to establish that the groove originates as Bradley described, with this difference that the place of origin is not the border of the cerebellum, but is situated between this border and the vermis. (See figure 12.)

VII. *Sulcus praetonsillaris sive sulcus inferior anterior.*

This fissure separates the tonsil from the lobus biventer. The general opinion is that it originates bilaterally, grows out medially and meets the fissura secunda (Elliot Smith, Bolk, Ingvar, Langelaan). At the same time, however, Ingvar states that the sulcus praetonsillaris does not

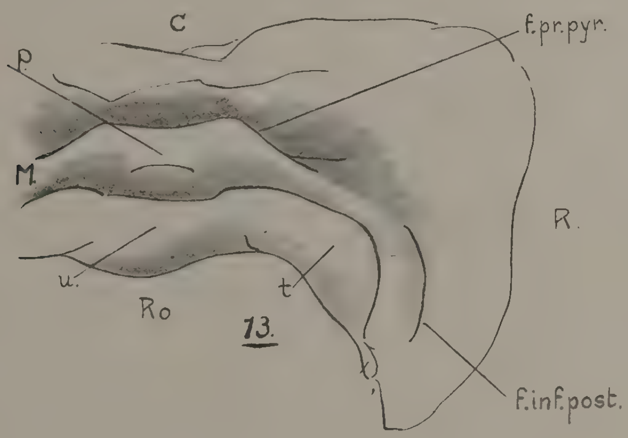
extend ar far as the border of the cerebellum, so that the tonsil remains still connected with the rest of the hemisphere. Our own observation confirms entirely with INGVAR's statement (see fig. 22).

VIII. *Fissura secunda.*

As for the development of this groove, there is general unanimity, namely, that it originates in the midline. It then develops laterally and becomes continuous with the sulcus praetonsillaris. We can only confirm this.

IX. *Sulcus praepyramidalis.*

All authors agree that this fissure makes its appearance in the midline shortly after the fissura secunda, extending laterally. A point of contro-



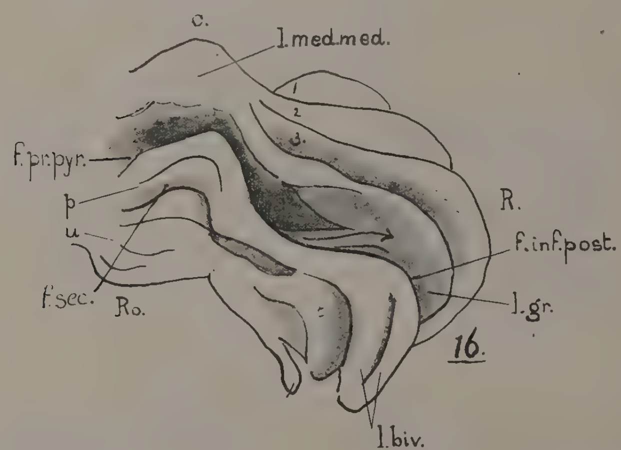
versy, however, is whether it passes over into the sulcus inferior posterior, or not. According to SMITH and LANGELAAN it does so. BOLK and JACOB regard the groove as the lateral elongation of the fissura praepyramidalis. INGVAR speaks of its merging into the fissura parafloccularis. Our observations are as follows: In fig. 13 is shown that the sulcus praepyramidalis terminates suddenly at some distance from the midline. This is a striking fact as the groove is fairly deep up to the end. More laterally there begins what we would term a secondary groove (see below). Laterally — albeit at some distance from the border — the sulcus inferior posterior begins, thus refuting BOLKs theory. In earlier stages — see fig. 14, plate II — it appears that the last-named groove runs caudally to the sulcus praepyramidalis and finally becomes continuous with a fissure of the pyramis, as can be seen in fig. 15.

In the depth of the groove which lies between the pyramis and the

lobus medius medianus, there is a lamella connecting the pyramis and the lateral part of the lobulus gracilis (fig. 16). The situation here represented is the continuation of that represented in figure 17, Plate III. From this it is evident that a part of the hemisphere lying dorsally to the sulcus inferior posterior, is connected with the pyramis. The sulcus praepyramidalis proper is thus but short, being "bridged" at some distance from the midline by the aforesaid lamella to what will be later the lobulus gracilis.

The situation thus arising is liable to lead to confusion and explains that there are various opinions regarding this problem.

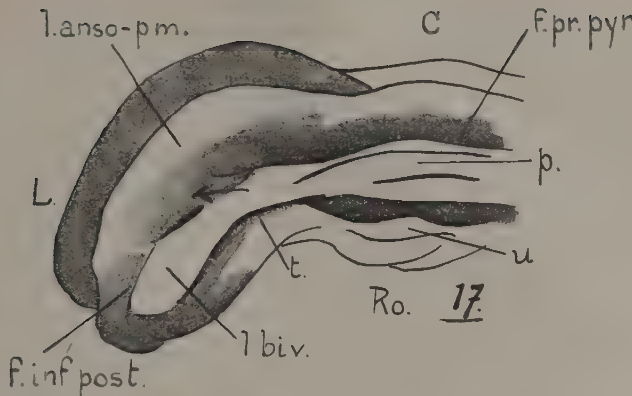
INGVARS conception, cited above, we cannot share, for the reason what this author regards as fissura paraflocculatis appears from the text to be actually the fissura flocculatis. This latter fissure is certainly not con-



tinuous with the sulcus praepyramidalis. Before going further into this matter it will be necessary, however, to explain the terms *primary* and *secundary* groove.

X. We hold the opinion that two kinds of fissures are to be distinguished on the cerebellum and that this differentiation is due to the mode of origin. We would designate as *primary* such a groove as is formed by two adjacent regions each of which later has a principally different function. Such one we term the sulcus uvulo-nodularis, the fissura prima and the sulcus praepyramidalis. The *primary groove* originates, as it were, by the evolution of two different regions which it separates. From their nature, all early formed sulci are primary. — The *secundary fissure*, on the contrary, separates regions with the same function. It appears later in the ontogeny, a part of this region developing more strongly than

PLAAT III.



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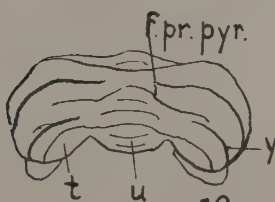

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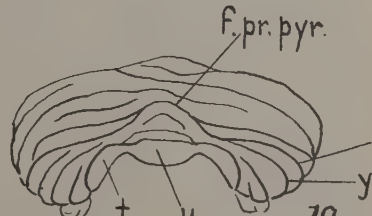

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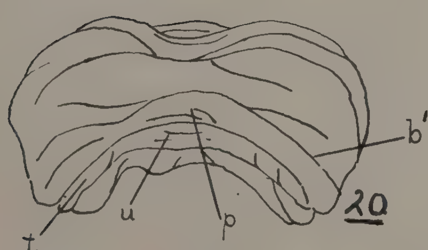

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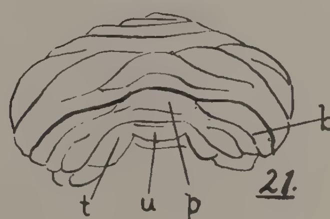

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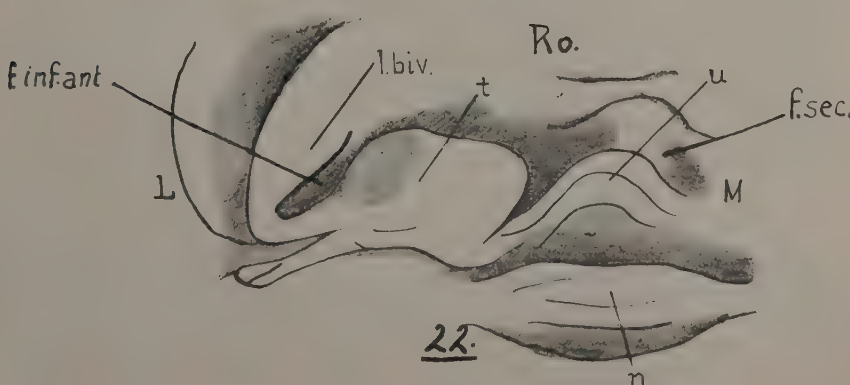

22.

the adjacent part, it is wide, shallow and not sharply demarcated from the surroundings. A *secundary groove* is, in our opinion, the sulcus paramedianus, which separates the vermis from the hemispheres and which originates by the excessive growth of the latter in connection with the first named.

By reason of the different genesis of the *primary* and *secundary* fissures. their course is also different. The *primary* fissures run transversally, the *secundary* sagittally. Although we do not ascribe to the secundary fissures the significance of a *primary* one, it may happen that when a *primary groove* passes into a *secundary* one, this creates the impression that the latter is a continuation of the former or, in other words, that there is an intimate association between them. This, however, is not the case, as is shown above.

XI. INGVAR mentioned that the sulcus praepyramidalis ends in an "Einkerbung" behind the "globular" cerebellar hemisphere. This is so indeed. But the said "Einkerbung" is clearly formed by the strong growth of the lobus ansoparamedianus, in other words, the "Einkerbung" is a *secundary fissure*. It is therefore not correct here to speak of the continuation of the sulcus praepyramidalis, as the last-named groove is a *primary* one. The same holds good for the sulcus parafloccularis (or properly speaking, the sulcus floccularis, see above) which according to INGVAR also ends in the "Einkerbung". We would rather compare this situation to a fissure in the earth, ending in a valley between two hills. To consider the valley as the continuation of the fissure would be an error.

XII. We should now consider the genesis of the human paraflocculus herein-after to be termed "Nebenflocculus HENLE". In LANGELAANS opinion ('19) this is the remnant of that part of the lobus uvulo-tonsillaris that atrophies owing to the pressure exercized by the developing hemisphere. Our observation, on the contrary, was that the secundary flocculus is attached to the stalk of the flocculus. Besides the sections of the cerebella of human fetuses showed us that only the flocculus and the "Nebenflocculus HENLE" had been myelinised. This was not the case with the cerebellar hemispheres. A strong fibre bundle connecting both flocculus and "Nebenflocculus" could also been detected.

From the situation thus described it is evident that the "Nebenflocculus" belongs to the flocculus and that the two form one [1]).

[1]) This is in agreement to what HAYASHI ('24) states, viz. that the flocculus consists of two parts: the flocculus sensu strictiori and „Nebenflocculus". Besides, several cases of atrophia neocerebellaris have been described, whereby the „Nebenflocculus" proved to be intact (BROUWER '12 and others). This may be an argument in favour of our view as described above.

XIII. If we now compare the development of the various fissures in the sub-primates with those as we described them in man, we can make the following statements:

1) The development and relations of the sulci uvulo-nodularis, praepyramidalis and of the fissurae secundae are the same in man as in the subprimates.

2) The human sulcus praetonsillaris develops in the same way as the sulcus intratonsillaris in lower mammals, the only difference being that the latter in the subprimates is first formed in the border of the cerebellum, whereas the human sulcus praetonsillaris can be observed first at some distance from this border.

3) What was sub 2 respecting the sulci praetonsillaris and intratonsillaris holds good also for the sulci inferior posterior and paraflocculаris, the only difference being again that the former in contrast to the latter, is first formed at some distance from the border of the cerebellum.

4) The relation of the sulcus praetonsillaris with respect to the fissura secunda is the same as that of the sulcus intratonsillaris with respect to this fissure.

5) The same holds good also for the sulci inferior posterior and parafloccularis with regard to the sulcus praepyramidalis.

With these points in mind we consider the sulcus praetonsillaris as being homologous with the fissura intratonsillaris and we hold the opinion that the fissura parafloccularis of subprimates is represented by the human sulcus inferior posterior.

The difference as regards the place of origin of these grooves, whether on the boundary of the cerebellum, or in the middle of the hemisphere, we do not consider of any decisive importance, considering the excessive development of the human hemispheres whereby it is very well possible that the place of formation may shift medially.

From the above it follows that the paraflocculus of subprimates is homologous with the human lobus biventer and the tonsil.

Abbreviations used in all figures.

C. Caudal
D. Dorsal
F. inf. post. Fissura inferior posterior
F. pr. pyr. Fissura praepyramidalis
F. pfl. Fissura parafloccularis
F. flocc. Fissura flocularis
F. sec. Fissura secunda
Floc. Flocculus
L. Left
L. anso. pm. Lobus anso-paramedianus
L. biv. Lobus biventer
L. gr. Lobus gracilis
Lat. Lateral
M. Median
N. Nodulus
Ne floc. "Nebenflocculus HENLE"
P. Pyramis
Pfl. Paraflocculus
Pf.d. Paraflocculis dorsalis
Pf. v. Paraflocculus ventralis
R. Right
T. Tonsil
Ro. Rostral
U. Uvula
V. Ventral

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(Communicated at the meeting of October 27, 1945.)

A. *Inleiding.*

Wanneer een gladde holle spier belast wordt, is in het algemeen de in de spier aanwezige spanning aanvankelijk onvoldoende om de last te dragen. De spanning neemt onder gunstige proefvoorwaarden geleidelijk toe en ten slotte wordt een evenwicht bereikt tussen last en spanning. De last wordt dan door de spier gedragen, deze handhaaft daarbij haar lengte. Wordt de spier nu geprikkeld, dan volgen een samentrekking en een ontspanning. Registreeren wij deze bewegingsverschijnselen op een draaiend kymografion, dan superponeren zich op een horizontale lijn de verkorting (uitslag naar boven) en de teruggang naar de rustlengte. Wij kennen zulke myogrammen ook van de reacties der dwarsgestreepte skeletspieren op een door hen ontvangen prikkel. Hetzelfde geldt in beginsel voor het mechanogram, dat wij kunnen registreeren van prikkelreacties van een gladde spier, *vóórdat* het evenwicht met de last is ingetreden. Zo lang er niet geprikkeld wordt, is het myogram een gebogen lijn, die aanvankelijk steil daalt en daarna steeds vlakker gaat verlopen, naarmate de spanning toeneemt. Dit mechanogram komt in uiterlijke vorm overeen met de z.g. BINGHAM-kromme, waarvan de betekenis door H. J. JORDAN en zijn medewerkers bij herhaling is uiteengezet (6, 7, 10 en 17).

Met betrekking tot de reactie op prikkeling van de slakkevoet, een holle gladde spier van een Evertebraat, heeft JORDAN twee vormen onderscheiden: de *spierschok* en de *tonische contractie*. De eerste bestaat uit een snelle samentrekking en onmiddellijk daarop volgende ontspanning, waarvan het myogram overeenkomt met dat der aan skeletspieren op te wekken schokken. Het verschijnsel superponeert zich echter niet op een horizontale rustkromme, doch op de BINGHAM-kromme. Met dit reactietype kan de slak echter niet volstaan. Er moet een contractie bestaan, waardoor de oorspronkelijke spiertoestand weer hersteld wordt, m.a.w. daarna moet een kromme te registreren zijn, die vanaf het niveau van de contractietop parallel loopt aan de kromme, welke aan de prikkeling voorafging. Dit wordt dus een kromme, die zich niet superponeert op de BINGHAM-kromme, maar daarvan een herhaling geeft (herhalingskromme).

[1]) Onderzoek verricht met steun van het Prov. Utr. Genootschap.

De spier heeft dan ook de rekkingsweerstand hersteld en daaraan voldoet de spierschok niet. Deze vorm van contractie, waarvan het bestaan theoretisch werd afgeleid (4), werd als tonische of langzame contractie ook naar haar wezen van de spierschok onderscheiden. De spierschok werd herleid tot de reactie van de slakkevoet in de toestand van een elastisch lichaam, de tonische contractie zou door stromingsverschijnselen binnen de spier als plastisch lichaam de oorspronkelijke toestand weer herstellen.

De Marees van Swinderen (18) kon door temperatuursschommeling een zodanige prikkeling uitoefenen, dat de langzame contractie volgde; de decrescente van het myogram leverde een nieuwe Bingham-kromme. waarop de door andere prikkeling opgewekte spierschokken zich superponeren. Jordan (5) kon bij Aplysia door snelle temperatuursverhoging (van 6° op 25° C) ook een langzame tonische contractie opwekken, die zich van de snelle spierschok duidelijk liet discrimineren. Men slaagde echter niet erin, de tonische contractie door electrische prikkeling op te wekken. Dat probleem moest dus nog tot oplossing worden gebracht, en wel om de volgende reden. Het dier is immers niet in staat door temperatuursschommelingen impulsen op te wekken; in de normale biologie kunnen die schommelingen dus geen doorslaggevende rol spelen. Electrische prikkels hebben daarentegen het voordeel, dat zij bij spier en zenuw verschijnselen veroorzaken, die het meest overeenkomen met die. welke door adaequate prikkels worden opgewekt, terwijl voorts ook de sterkte der electrische prikkeling op eenvoudige wijze is te regelen.

N. Postma vond bij zijn tonus-onderzoek het eerst enige factoren, die het hunne bijdragen tot het optreden van een tonische contractie: de spier mag tijdens de contractie niet belast zijn (12) en de faradische prikkeling mag niet te sterk zijn en niet te lang duren (11, p. 65). J. A. Maas (10) constateerde vervolgens, dat zelfs een geringe uitwendige weerstand (o.m. de wrijvingsweerstand in de registratie-apparatuur), door de slakkevoet tijdens de samentrekking ondervonden, storend werkt. Hij verving verder de electrische prikkeling door mechanische, n.l. door met de punten van een stomp pincet over de flanken van de slakkevoet te strijken. Van deze zelfde prikkeling gebruik makende, kon Postma (14) nog vaststellen, dat de grootte van de reklast tijdens de aan een contractie voorafgegane rekking niet onverschillig is: slechts een z.g. indifferente last waarborgt bij reflectorisch opgewekt tonusherstel een congruente herhaling van de Bingham-kromme. Voor wat de invloed van prikkelsterkte en -duur betreft, was het probleem nog steeds niet geanalyseerd; mij werd nu opgedragen dit nader te onderzoeken.

Bij de desbetreffende proeven konden wij aansluiting zoeken bij ervaringen, die Postma had opgedaan bij prikkeling van zenuwcentra (16) en van de voetzenuwen (14, 15). Het onderzoek werd geheel in overleg met hem uitgevoerd; voor de van hem ondervonden hulp en belangstelling moge hier hartelijke dank worden betuigd.

B. *Probleemstelling en methodiek.*

De BINGHAM-kromme, verkregen van een belaste slakkevoet, is een rekkingskromme. De helling daarvan geeft een indruk van de tegen de rekking geboden weerstand; naarmate de kromme steiler loopt, is er een groter verschil tussen de aanwezige spanning of tonus en de reklast. JORDAN (3) heeft gevonden dat die tonus geregeld wordt door het pedaalganglion. POSTMA (16) stelde vast, dat die regeling geschiedt door middel van impulsen, welke door het tonuscentrum langs de pedaalzenuwen naar de voet worden uitgezonden. Daarbij zijn te onderscheiden tonusverlagende en tonusverhogende impulsen. POSTMA heeft deze met behulp van electrische prikkeling kunnen opwekken.

Bij prikkeling van het pedaalganglion, terwijl de spier door een constante last wordt gerekt, krijgen wij volgens POSTMA de volgende verschijnselen: Bij zeer zwakke, subliminale prikkels gebeurt er niets. Wordt echter een drempelwaarde overschreden (fig. 3, R.A. = klosafstand: 30 cm), dan treedt weerstandsverlies op. Met voortschrijdende versterking van de prikkel neemt de verlaging van de weerstand toe, totdat klaarblijkelijk een nieuwe drempel (fig. 3, R.A.: 28 cm) wordt overschreden. Dan treedt een biphasische reactie op, beginnende met een kleine contractie, welker decrescente dieper daalt dan het voetpunt waarbij de crescente begon; hieruit blijkt dus een nakomend tonusverlies. Met nog verder toenemende prikkelstrekte neemt de contractie toe evenals de tonusverlagende werking, terwijl deze laatste vanaf een zekere intensiteit (fig. 3, R.A.: 26 cm) weer geringer wordt. Tenslotte verdwijnt de doorzakking geheel en houden wij alleen een op de rekkingskromme gesuperponeerde contractie over. Men verwekt dus bij een zwakke prikkeling tonusverlies, bij sterke prikkeling kortstondige contractie, en bij prikkeling met matige sterkte een *combinatie* van beide verschijnselen.

Prikkeling van de zenuwstammen, welke van het pedaalganglion naar de voet lopen, geeft een zelfde verband tussen prikkelsterkte, tonusverlagende werking en opwekking van contractie te zien. Het mechanogram geeft echter vaak bij toenemende prikkelsterkte het volgende beeld: Eerst geen verstoring van het verloop der rekkingskromme, d.w.z. de prikkel is subliminaal voor alles. Na overschrijding van de drempel volgt weerstandsverlaging, die eerst toe- en daarna weer afneemt, vervolgens een intensiteits-gebied, dat „niets" oplevert en ten slotte contractie. Dat „niets" is slechts schijnbaar rust en komt tot stand, doordat de beide tegenovergestelde reacties elkaar opheffen. Er is dan een *summatie* in plaats van combinatie der effecten. Waar JORDAN (3, p. 579/80) schreef met toenemende prikkelsterkte „niets" òf contractie te verkrijgen, had hij stellig met dit evenwichtsgebied te doen. Eerst POSTMA heeft dus van tonusverlagende of tonolytische impulsen bij het onderzoek gebruik gemaakt. Tijdens zijn onderzoek betreffende de tonische contractie werd de mogelijkheid van het opwekken dier tonolytische werking echter nog

niet voldoende beheerst. POSTMA leidde toen de prikkelstroom direct door de voet. Het ging er nu dus om, in het bijzonder na te gaan, of dezelfde afhankelijkheid van prikkelsterkte en effect ook is aan te tonen bij prikkeling van het perifere (in de voet gelegen) zenuwnet. Zoals gezegd, was ons probleem dus: *de invloed na te gaan van de intensiteit der Faradische prikkeling op de periferie (voet en zenuwnet) bij het opwekken van de tonische contractie.*

Wij maakten bij ons onderzoek gebruik van de herhalingskromme (POSTMA 11, 12, MAAS 10). Nadat de eerste rekkingskromme opgenomen is (de z.g. originele kromme, „O.K.'' MAAS) met een reklast, die naar schatting de indifferente last zo goed mogelijk benadert, wordt de reklast verwijderd. Vervolgens wordt de schrijver opgetrokken, zodat de door reflectorische (mechanische) prikkeling op te wekken contractie zich kan voltrekken, zonder dat uitwendige weerstand wordt ondervonden. Is door de samentrekking de draad naar de schrijver weer gestrekt, zodat de spier zijn oorspronkelijke verkortingsgraad heeft bereikt, dan wordt opnieuw met de eerst gekozen last gerekt, telkens gedurende 10 minuten; dan registreren wij de eerste herhalingskromme. Wijkt deze duidelijk van het verloop der oorspronkelijke kromme af, dan wordt voor de volgende herhalingskromme een andere reklast genomen: kleiner dan de eerste indien de kromme steiler liep, groter in geval van een vlakker verloop. Hebben wij door een gelukkige keuze direct de indifferente last te pakken, dan blijkt dat uit de congruentie der eerste twee herhalingskrommen onderling en met de originele kromme. Behoudens geringe schommelingen, vertonen de onderlinge verschillen geen gang, die zich met de rangorde der krommen in een bepaalde richting — hetzij als tonusverlies, hetzij als tonustoename — aftekent. Is de indifferente last op deze manier gevonden, dan voltrekt zich dus telkens de volkomen tonische contractie, die behalve de verkortingsgraad ook de weerstand tegen rekking herstelt.

Daarna kan dan — bij handhaving van de gevonden indifferente last — de mechanische prikkeling vervangen worden door Faradische, waarvan de sterkte achtereenvolgens wordt opgevoerd door verschuiving van de secundaire inductieklos naar de primaire. Er wordt dus begonnen met een grote klosafstand (R.A.), d.w.z. zwakke prikkeling. Daar de voetspieren individueel zeer verschillend reageren op de prikkelsterkten, doordat de drempelwaarde grote verschillen vertoont, hangt er veel af van het geluk, of de gekozen sterkte juist geen verandering van de tonische toestand blijkt te geven. Wij krijgen dan met Faradische prikkeling na het optrekken van de schrijver even goed tonische contractie als door de mechanische prikkeling. Blijkt de tonus wèl gewijzigd te zijn — wijkt de herhalingskromme dus af van die met indifferente last en mechanische prikkeling — dan moet nog worden gecontroleerd of de verandering een gevolg is van een prikkeleffect op de tonus zelve, dan wel van een drempelverschuiving ten opzichte van de indifferente last. Tijdens de reeks van waarnemingen

wordt daarom na een door Faradische prikkeling opgewekte verandering van tonus (en na de op de contractie volgende rekking) weer mechanisch geprikkeld, om te zien, of de wijziging zich niet voortzet. Hadden wij b.v. met tonusvermindering te doen, dan bleek deze meestal weer terug te gaan; soms werd de oorspronkelijke weerstand geheel hersteld. In zulke gevallen mag de bij Faradische prikkeling optredende verandering van rekkingsweerstand inderdaad aan die prikkeling worden toegeschreven. Alleen op den duur trad tonusverlies op of kwam — vooral bij sterke prikkeling — daarvoor contractuur in de plaats, welke verandering zich ook voortzette bij aanwending van reflectorische prikkeling en met gebruik van de last, die eerst „indifferent" was. Klaarblijkelijk verouderde het proefobject te veel, ten gevolge waarvan het dan niet meer bruikbaar was.

Met betrekking tot het gebruik van geringe prikkelsterkten valt op te merken, dat daarbij alleen samentrekking en gelijktijdig tonusverlies zouden zijn te verkrijgen, indien de prikkeldrempels voor contractie en tonolyse niet voor alle elementen van ons proefobject gelijk zouden zijn. Inderdaad bleken deze elementen zich als het ware hetero-liminaal te gedragen. Bij zeer geringe prikkelsterkten moet langer geprikkeld worden dan bij grotere, om door middel van haar contractie-verwekkende werking tenslotte de oorspronkelijke verkortingsgraad van de spier hersteld te krijgen. Langdurige prikkeling stompt echter af; om dit te voorkomen werd de prikkeling, zo nodig enige malen, om de seconde onderbroken.

Bij het oudere onderzoek werd de prikkelstroom toegevoerd door stanniolreepjes, waarover de spier werd gelegd (JORDAN 8, DILLEWIJN en 's JACOB 1), of door spelden, waarmee de voet in de registratie-apparatuur werd bevestigd (HERTER 2, POSTMA 11). Wij hebben de stroom niet overlangs door het object geleid, maar in verticale richting. De ene electrode werd gevormd door het met stanniolpapier bedekte rektafeltje met aansluiting t (Fig. 1); de andere electrode bestond uit twee evenwijdige beugels van zilverdraad (b_1 en b_2), die ieder op een flank van het proefobject worden geplaatst. De electrodehouder f is daartoe met behulp van een micrometerschroef verstelbaar in verticale richting.

Ideaal zou het natuurlijk zijn, als aan één en hetzelfde object de gehele schaal van prikkelsterkten getoetst zou kunnen worden. Dit bleek echter onmogelijk, daar de proefobjecten te veel verouderen tijdens de waarnemingen. Immers iedere rekking vordert 10 min en de verwijdering van de reklast, het optrekken van de schrijver, alsmede de opwekking van de tonische contractie minstens 3 min, elke opname dus ten minste 13 min. De bepaling van de indifferente last geschiedt aan minstens drie krommen, soms echter aan zeven. Daarna vergt iedere opname na electrische prikkeling en de daarbij behorende contrôle met reflectorische ongeveer 30 min. De bepalingen van de indifferente last en van een drietal punten der intensiteitsschaal eisen dus gemiddeld rond 150 min, zodat

het praeparaat dan ondertussen bijna 3 uur oud is. Wij moesten daarom met meerdere objecten werken, doch waar de voetspieren individueel zeer verschillend reageren op de prikkelsterkte, diende een methode te worden gevonden om de resultaten der waarnemingen, verricht bij verschillende voeten, tot een bepaalde basis te herleiden, teneinde quantitatief vergelijkbare gegevens te kunnen verkrijgen, die elkaar tot een volledige reeks zouden aanvullen.

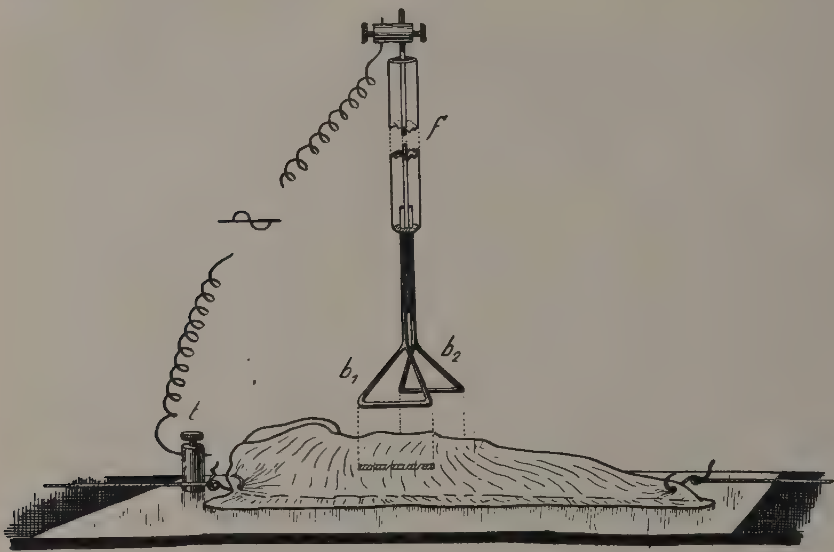


Fig. 1.

Schets van de electroden voor Faradische prikkeling, welke electroden de stroom direct op de voetspier en zijn zenuwnet overdragen. Overige toelichting in de tekst.

Daartoe werd voor elk proefobject de bij de indifferente last met reflectorische prikkeling verkregen herhalingskromme als basis genomen. Voor de berekeningen bezigden wij het gewicht van het papieroppervlak, dat omsloten werd door de abscis (aan de bovenzijde), de ordinaat (op de rechter zijde) en de kromme van links-boven naar rechts-onder (vgl. N. Postma 13, p. 1154). Dat oppervlak werd op 100 gesteld en de veranderingen in het verloop der krommen werden daarop omgerekend. Bij verhoging van de tonus wordt het oppervlak dus kleiner dan 100, bij tonolytische werking der prikkels groter dan 100.

Bij de Faradische prikkeling werd altijd met grotere klosafstanden begonnen en vervolgens de prikkelsterkte verhoogd door de afstand kleiner te maken.

C. *Resultaten en bespreking.*

Het kwam er nu dus op aan na te gaan, of met toenemende prikkelsterkte achtereenvolgens waar te nemen waren: 1e. toenemend weerstands-

verlies, met een maximum, waarna een vermindering daarvan zou verschijnen; 2e. eventueel een overgangsgebied, waar de kromme zich handhaaft op een waarde van omstreeks 100 %; en 3e. een toenemende verhoging van de tonus. Inderdaad werden verschillende delen van deze

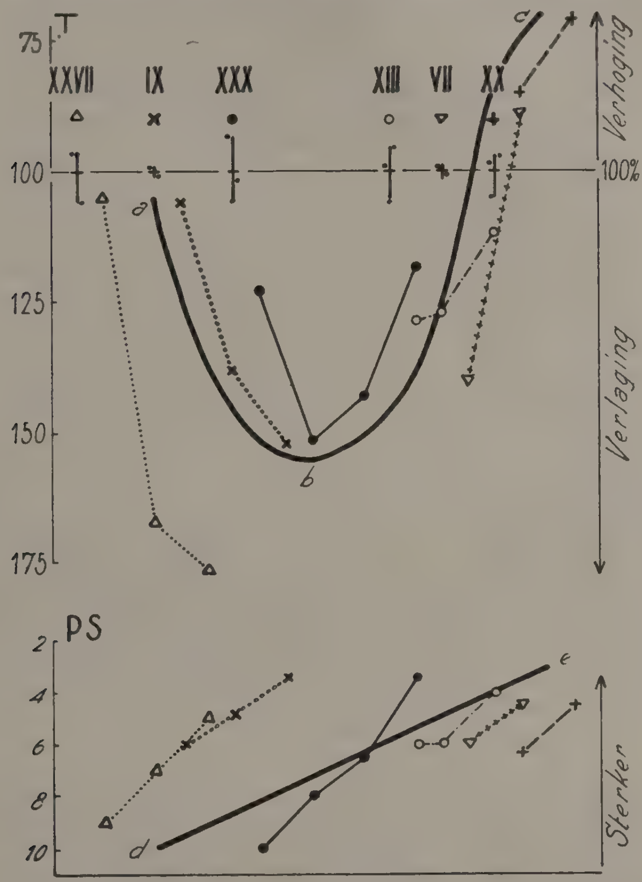


Fig. 2.

De bovenste figuur geeft de tonus-verlaging of -verhoging; T = tonus; 100 % = tonus ongewijzigd. De onderste figuur geeft de bijbehorende prikkelsterkten (= PS) in cm afstand tussen secundaire en primaire inductieklos. De dik getrokken lijnen a-b-c en d-e zijn afgeleide krommen. Overige toelichting in de tekst.

reeks van effecten aan onderscheidene voeten gevonden. Tabel I geeft een overzicht van het geheel der resultaten. In fig. 2 zijn enige hiervan in een aaneensluitende reeks tot een grafische voorstelling samengesteld. Beschouwen wij de waarden, verkregen van object XXVII, bij wijze van voorbeeld nader:

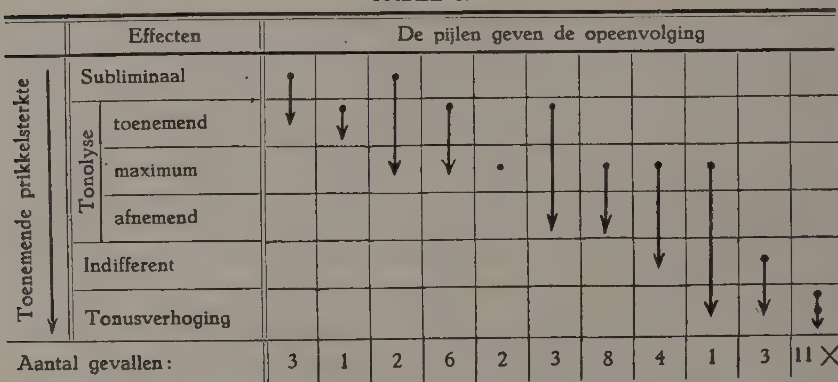
De drie herhalingskrommen (opgenomen met de voor dit object indifferente last) leverden 218,75, 218,75 en 239,5 mg, dus gemiddeld

225,5 mg, hetgeen op 100 % gewaardeerd wordt. Daarop omgerekend krijgen wij dan voor de herhalingskrommen 97, 97 en 106 %; deze waarden zijn links bovenaan aangegeven. Zo is op de abscis van 100 % voor ieder object de speling te beoordelen van de herhalingskrommen, opgenomen na herstel door reflectorische prikkeling. Vervolgens werd de tonische contractie opgewekt door Faradische prikkeling; in de onderste helft van de grafiek zijn de prikkelsterkten opgegeven in cm klosafstand Zo vonden wij voor nummer XXVII na elkaar:

Klosafstand	9 cm:	oppervlak	106 %,	contrôle	refl.pr.	108 %;
,,	7 ,,	,,	168 %,	,,	,,	152 %;
,,	5 ,,	,,	176 %.			

De eerste Faradische prikkeling was dus nog subliminaal, de tweede werkte sterk tonolytisch, de derde voegde daaraan nog iets toe.

TABEL 1.



Zo werden naast elkaar de uitkomsten van de objecten XXVII, IX, XXX, XIII, VII en XX in grafiek gebracht en leverde iedere spier een deel van het schema. De verschillende reacties hebben echter één beginsel gemeen, namelijk dat met toenemende prikkelsterkte een bepaalde volgorde der verschijnselen optreedt, hetgeen duidelijk in tabel 1 is te zien, indien men van de pijlen naar links opzoekt, welke effecten aan verschillende objecten achtereenvolgens werden verkregen. De meest overtuigende combinatie geeft wel de middelste pijl met de omslag in de tonolyse (3 maal waargenomen, w.o. object XXX afgebeeld in fig. 2). Daarom is het geoorloofd bij de samenstelling der grafiek die volgorde in acht te nemen. Dan tekent zich uit het geheel der waarnemingspunten een kromme a-b-c af, zoals die van één object verkregen zou kunnen zijn, indien de gehele reeks waarnemingen daaraan uitvoerbaar was. De bijbehorende lijn der prikkelsterkten is dan weer te geven met de kromme d-e. Wij vonden dus door prikkeling van de slakkevoet zelve — evenals Postma bij prikkeling van de pedaalzenuwen — twee intensiteits-gebieden, waarbij de

Faradische prikkeling geen verandering van de tonische toestand ten gevolge heeft. Bij de grotere klosafstanden is de prikkel klaarblijkelijk subliminaal, bij de sterkere prikkeling heffen tonusverlaging en -verhoging elkaar op, zodat de summatie een evenwicht oplevert. Wij kunnen een dergelijk gebied van prikkelsterkten ook verwachten. Immers, zetten wij de door POSTMA bij prikkeling van het tonuscentrum verkregen *combinatie* der tegengestelde effecten op een assenstelsel uit, zoals fig. 3 ze weergeeft, dan was er bij klosafstand 25¾ cm een gebied, waar de contractie een zelfde uitslag naar boven gaf als de tonolyse-naar beneden. Prikkelt men het centrum, dan treden beide reacties na elkaar op en zijn zij als zodanig te onderscheiden. Prikkeling van de periferie wekt ze klaarblijkelijk simultaan op. POSTMA vond bij prikkeling van de pedaalzenuwen beide reactie-typen, bij het ene object successief, bij het andere simultaan.

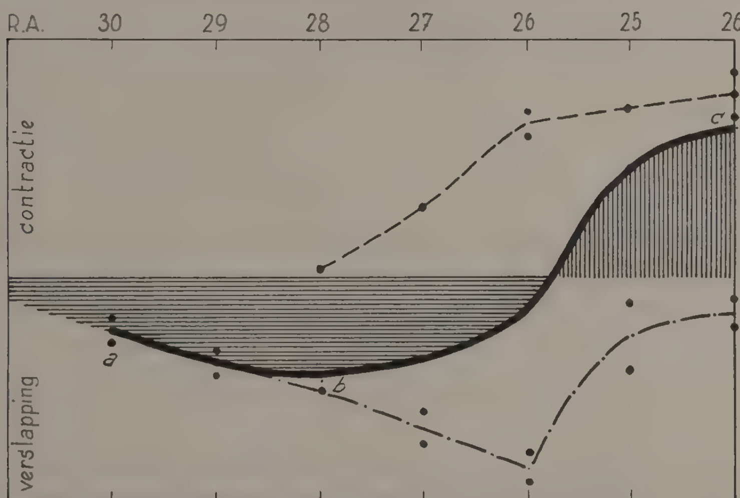


Fig. 3.

Krommen, afgeleid uit Abb. 3 der publicatie van JORDAN en POSTMA (9), p. 1174. Op de abscis de prikkelsterkten in cm klos-afstand (= RA), op de ordinaten de tonusverlaging naar beneden, de contracties naar boven. De zwaar getrokken kromme a-b-c geeft de algebraïsche summatie van de krommen, welke door de tonolyse-punten en door de contractie-punten getrokken kunnen worden. Die kromme stemt principieel overeen met de gefingeerde kromme uit fig. 2.

D. *Samenvatting.*

Wij onderzochten de invloed van de prikkelsterkte op het tonusherstel, verkregen door middel van een tonische contractie, opgewekt door Faradische prikkeling van de slakkevoet zelve.

Het mechanogram geeft bij toenemende prikkelsterkte achtereenvolgens: eerst niets (d.w.z. de prikkel is subliminaal voor alles), daarna tonusver-

laging, vervolgens niets en ten slotte tonusverhoging. Dat tweede „niets" is slechts schijnbaar rust en komt tot stand, doordat de beide tegenovergestelde reacties elkaar opheffen (summatie). Hierdoor is het verklaarbaar, dat bij opwekking van tonusherstel door electrische prikkeling na rekking in sommige gevallen een juist herstel werd verkregen (POSTMA 1935), en in andere gevallen òf verlies òf verhoging van de weerstand, als de samentrekking de verkortingsgraad hersteld had.

Wij vonden dus een zelfde verband tussen prikkelsterkte en prikkeleffect, als POŠTMA heeft waargenomen bij prikkeling van het pedaalganglion en van de pedaalzenuwen.

Zusammenfassung.

Wir untersuchten den Einfluss der Reizstärcke auf die Wiederherstellung des Tonus, wenn die tonische Kontraktion mittels Faradischer Reizung des Helixfusses ausgelöst wurde. Wir fanden die gleiche Gesetzmässigkeit wie POSTMA bei Reizung der Pedalganglien und der Pedalnerven: Tonuslösung mit eben überschwelliger Reizung, Tonuserhöhung mit Reizen grösserer Intensitäten und dazwischen täuscht das Mechanogramm Ruhe vor, weil die antagonistischen Effekte der „indifferenten" Reizung einander aufheben (algebraische Summation).

Abb. 1. Aufstellung für Faradische Reizung des Schneckenfusses. Zur einen Elektrode dient der mit Stanniolpapier bedeckte Tisch des Dehnungsapparates; der Reizstrom erreicht hier die Sohle des Muskels. Die andere Elektrode endet in zwei paralleler Bügel (b_1 und b_2) und wird in ihrem Halter (f) mittels einer Mikrometerschraube auf den Rücken des Fusses gesenkt. Der Reizstrom passiert den Fuss also in vertikaler Richtung.

Abb. 2. Grafische Vorstellung der Tonusänderungen, ausgelöst durch Faradische Reizung des Muskels und seines Nervennetzes. Die Resultate verschiedener Füsse sind mit römischen Ziffer und den darüberstehenden Merkmalen angedeutet. Der Tonus wurde gemessen an den Oberflächen zwischen Dehnungskurve, dem Ordinat des Endpunktes und oberer Abszisse. Pro Fuss diente die Fläche der Wiederhohlungskurven (der Fuss wurde zuerst mechanisch statt elektrisch gereizt) als Urmaass, dessen Mittelwert gleich 100 gesetzt wurde. Mittels diesen Urmaassen waren auch die Resultate verschiedener Füsse vergleichbar. Flächenvergrösserung durch Tonussenkung liefert dan mehr als 100, Tonuserhöhung weniger. Man findet in der oberen Hälfte der Figur einige Beispiele abgebildet. Der untere Teil gibt die zugehörigen Reizstärcken in cm Rollabstand an. Die dick gezeichneten Kurven a-b-c und e-d geben, die wircklich erhaltenen Daten berücksichtigend, wieder, wie ein Fuss reagieren würde, wenn mit steigenden Reizstärcken die ganze Intensitätsskala an einem einzigen Muskel durchgemessen werden könnte.

Abb. 3. Diese Kurven sind abgeleitet aus Abb. 3 einer Publikation von JORDAN & POSTMA (9, S. 1174). Die Abszisse gibt die Reizstäcken in cm Rollabstand. Auf den Ordinaten nach unten Tonussenkung, nach oben Kontraktion. Die dick ausgezogene Kurve a-b-c gibt die algebraische Summation der Lösungs- und Kontraktionskurven. Sie entspricht prinzipiell der fingierten Kurve a-b-c der Abb. 2.

Summary.

Our problem was the relation between the strength of the stimulation and the restauration of the tonus by tonic-contraction which is evoked faradically in the foot-muscle of the snail and its nerve-net. We found the same connection as POSTMA stimulating the pedal-ganglion and the pedal-nerves: Tonolyse with just-liminal stimulation, tonus-enchancement with stronger stimuli. Between these intensities of weak and strong excitation there are "indifferent" stimuli which produce no effect in the myomechanogram: it seams that they maintain rest, in fact the antagonistic effects compensate each other.

Fig. 1. Arrangement for faradic stimulation of the foot muscle of the snail. The stanniol-cover of the extension table is one electrode; here the alternating current reaches the sole of the foot. The other electrode ends in two parallel bows (b_1 and b_2). By means of a micrometerscrew its holder (f) can be placed downwards on the back of the foot muscle. Thus the alternating current passes the muscle in a vertical direction.

Fig. 2. Scheme of the tonus alterations evoked by direct faradic stimulation. The results obtained from different feet are marked with Roman numerals and the figures placed above the ciphers. Tonus is measured on the area enclosed by the extension curve, the ordinate of its end and the upper abscissus. Per foot muscle the area of some (at least three) consecutive repetitive graphs obtained by subjecting the foot to mechanical stimuli in stead of electrical ones, gives the standard: the average of these areas is fixed at 100. The standards of different feet enable us to compare the results obtained from these muscles. Tonus decrease is shown by increase of the area (more than 100), tonus increase by the opposite. The upper part of the figure shows some examples. The lower part gives the excitation-strengthes belonging to the data of the upper part. The curves a-b-c and d-e are constructed after the represented data. These graphs give an impression how a foot will answer to stimuli of increasing strength if it should be possible to test one muscle with all intensities.

Fig. 3 is derived from Abb. 3 in a paper of JORDAN & POSTMA (9, p. 1174). Absciss: incraesing excitation intensities (in cm coil distance). Ordinate: downwards means tonolyse, upwards contraction. The curve a-b-c is constructed by algebraic summation of the tonolyse and contraction graphs and is congruent with curve a-b-c in fig. 2.

Résumé.

Nous avons déterminé l'influence de l'intensité de l'excitation sur la restauration du tonus par une contraction tonique provoquée en excitant au moyen d'un courant inducteur le pied de l'escargot et son réseau nerveux. Nous avons trouvé une relation identique à celle que POSTMA a constatée en stimulant le ganglion pédal ou les nerfs pédieux: Tonolysie avec des stimulus justement liminairs, tonogenèse par excitation plus intense. Entre ces intensités de stimulation faible et forte, on trouve des excitants moyens qui ne causent aucun mouvement dans le méchanogramme enregistré. En apparence ils n'attaquent pas l'état de repos du muscle, en réalité les effets antagonistiques s'équilibrent.

Fig. 1. Arrangement pour la stimulation faradique. La couverture de papier d'étain sur la table de traction sert d'électrode. Elle apporte les courants induits à la sole du pied. L'autre électrode dont le porteur se meut verticalement par le moyen d'une vis micrométrique se termine par deux cuillers parallèles (b_1 et b_2). Elles sont placées sur le dos du pied. Le courant induit traverse donc le muscle en direction verticale.

Fig. 2. Schéma des modifications du tonus provoquées par la stimulation du pied. Les résultats obtenus sur différents pieds sont marqués de chiffres romains et distingués par les signes placés au-dessus des chiffres. Le tonus est mesuré à la superficie limitée par la courbe d'allongement, l'ordonné de la fin de cette courbe et l'abscisse supérieure. Pour chaque muscle on trouve un témoin valide dans la grandeur moyenne des superficies appartenant à plusieurs (au moins trois) courbes successives. Celles-ci sont obtenues après des contractions toniques, provoquées par stimulation mécanique au lieu de stimulation électrique. La moyenne est fixée à 100. Ce témoin individuel peut aussi nous servir de base pour comparer les résultats obtenus sur différents muscles. La diminution du tonus est signalé par un accroissement de la superficie au delà de 100. L'augmentation du tonus produit l'inverse. La moitié supérieure de la figure donne quelques exemples. La partie inférieure montre les intensités des excitants faradiques en cm distance de la bobine induite; elles sont reliées aux résultats, donnés au-dessus. Les courbes a-b-c et d-e sont construites en rapport avec les exemples donnés. Elles nous donnent une idée approximative des réactions du muscle à des stimulus d'intensité progressive, s'il était possible de parcourir toute l'échelle d'intensité avec un seul pied sujet.

La fig. 3 est dérivée de la fig. 3 dans l'article de JORDAN & POSTMA (9, p. 1174). Abscisse: acroissement de l'intensité du stimulus. Ordonnés: du bas indiquent tonolysie, ceux du haut la contraction. La courbe a-b-c est construite par l'addition algébrique des courbes de tonolysie et de contraction; elle est équivalent à la courbe a-b-c de la fig. 2.

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Comparative Physiology. — *De la plasticité du pied de l'escargot (Helix pomatia Linné) et de ses types de raccourcissement et de rallongement* [1]). By N. POSTMA. (Communication du Laboratoire de Physiologie comparée de l'Université d'Utrecht.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of October 27, 1945.)

A. *Introduction.*

Après la série septénaire d'articles de JORDAN, de DE JONG et de nous-mêmes (10, 11, 12, 15, 16 et 6) dans laquelle nous avons décrit les réactions toniques du pied de l'escargot, le moment est venu d'envisager à nouveau la plasticité de ce muscle du point de vue de la physiologie comparée.

Cette plasticité est une propriété importante du tonus musculaire. Le phénomène consiste en ce que le muscle — pourvu que certaines conditions soient réalisées — comme une masse modelable, se laisse imposer n'importe quelle altération de longueur. Pour étudier la réaction plastique au muscle strié squelettique des Vertébrés, il est nécessaire de laisser le nerf moteur de ce muscle *en connexion avec le système cérébro-spinal.* Probablement grâce aux liaisons neuro-myo-épithéliales chez les animaux inférieurs (Actinies, Mollusques) et aux articulations synaptiques périphériques des viscères (l'estomac, l'utérus, la vessie, etc.) les muscles *lisses* avec des réseaux nerveux exhibent une réaction plastique *malgré leur isolement du système* nerveux central (céphalo-rachidien). — La *façon de réagir* plastiquement est différente aussi:

Si nous essayons de fléchir à l'articulation du genou la patte postérieure plus ou moins étendue d'un chat décérébré, les extenseurs opposent de la résistance jusqu'à ce que la force exercée menace de faire déchirer les muscles. Alors la jambe inférieure cède *soudainement* à la pression, jusqu'à ce que celle-ci devienne insuffisante: la nouvelle attitude plus fléchie est gardée. Il n'y a pas de tendance à reprendre la position initiale comme nous pourrions nous y attendre, vu l'action combinée de l'élasticité et du tonus des extenseurs. De même la jambe inférieure se laisse étendre passivement. *Ces mouvements sont extraordinairement brusques.* Avant que les extenseurs cèdent, des réflexes myotatiques ajustent le tonus des muscles à la force de pression progressive jusqu'à ce que le seuil d'un réflexe inhibitif soit franchi. Puis la résistance est *totalement* supprimée et ce freinage se maintient aussi longtemps que la pression pratiquée est supérieure au seuil de l'inhibition. Si la force

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[1]) Ce travail a bénéficié de l'aide de l'Assoc. d. l. Prov. d'Utr. pour les Arts et les Sciences „Utr. Prov. Genootsch. v. Kunsten en Wetensch.".

exercée descend au-dessous de cette limite, les réflexes myotatiques fixent de nouveau la patte. Les différences de tension musculaire se manifestent donc par sauts brusques, également quand *l'extension* est imposée (*réactions d'allongement* et *de raccourcissement* de SHERRINGTON 17, 18). Le jeu alterné de ces deux réactions détermine la plasticité du tonus. Ces mouvements plastiques des muscles du chat décérébré sont *assouplis par des réflexes supérieurs*. Nous remémorons l'inhibition cérébrale constante et l'action dynamogénique contenue des centres entre la moelle épinière et le cerveau, avec les réflexes de posture et d'équilibre du corps, assurés par les liaisons intercentrales du cerveau, des couches optiques, des corps striés et du cervelet. — Au contraire le muscle strié *isolé du système nerveux central* montre une élasticité parfaite: après sa distension passive, il revient entièrement à son état primitif. On distingue aussi le comportement plastique de la préparation neuro-musculaire centrale sous le nom d',,élasticité faible'' [2]).

Chez les muscles lisses des organes creux, l'antagonisme fixé par les rapports anatomiques tels que nous les connaissons dans les muscles squelettiques fait défaut. L'altération de longueur de ces muscles leur est imposée par l'augmentation de volume ou le déplacement du contenu des organes (l'assimilation alimentaire, la croissance embryonnaire, le remplissage de la vessie). Donc ces muscles creux sont exposés à une traction et ils cèdent à l'allongement de sorte *que la force exercée doit vaincre constamment une certaine résistance*. D'autre part, la tension permanente est insuffisante pour résister subitement à une augmentation de la pression interne. Le tonus s'adapte *progressivement* à la force de traction par le moyen du muscle qui n'a plus de relation avec le système nerveux végétatif. La réaction d'allongement fait penser à celle de la matière plastique dans la nature inanimée. Cette réaction est caractérisée par un changement graduel de longueur; la pression diminuant, le muscle se raccourcit peu à peu.

Le sac musculeux lisse est *susceptible* de réagir plastiquement d'une façon indépendante tandis que le muscle squelettique est subordonné au système nerveux rachidien. Les réactions plastiques périphériques des muscles lisses sont accélérées par *les centres supérieurs;* ceux-ci *favorisent donc les extrêmes:* En premier lieu la résistance à l'allongement est abolie; d'autre part, l'extensibilité est supprimée de sorte que le muscle résiste *totalement* à la force de traction et qu'il se comporte comme un corps parfaitement élastique. On peut comparer approximativement cette réaction avec celle d'une bulle de savon très extensible.

A l'inverse de ceux du muscle squelettique, les mouvements tolérés par les muscles lisses qui se trouvent en connexion avec le système nerveux central (chez les Vertébrés, le grand sympathique) sont plus brusques

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[2]) E. GLEY, Traite élémentaire de Physiologie, T. II, p. 981.

que ceux de l'objet musculeux périphérique. Dans ce cas, *les mouvements rudes de la périphérie sont assouplis* par des réflexes supérieurs qui modèrent consécutivement la résistance et font réagir l'objet comme un muscle à cliquet.

Dans le système somatique des Vertébrés, les impulsions primaires — qui déclanchent la contraction des unités musculaires — et les influx inhibitifs sont conduits *en amont* le long des voies *afférentes* à l'axe cérébro-spinal. *En aval, les influx accélérants* seuls sont portés secondairement par des axones efférents aux entités motrices du muscle. Ces ensembles de machines musculaires élémentaires sont composés de fibres innervées par le même cylindre-axe (SHERRINGTON). Le nombre des unités motrices excitées est déterminé par le nombre d'impulsions secondaires dont dépend aussi le degré de contraction (tétanos). A son tour, le nombre de ces influx dépend du rapport entre les influx accélérants primaires et les impulsions inhibitives. Une baisse du nombre des influx moteurs secondaires aboutit à une démobilisation d'entités musculaires et cause une diminution du tonus.

Les muscles lisses des viscères reçoivent à la fois du système nerveux central autonome *le long des nerfs efférents des impulsions accélérantes et démobilisantes* (CL. BERNARD). C'est pourquoi ces muscles sont capables de réagir plus promptement que sans de tels influx centraux inhibitifs. Et dans la périphérie, le même antagonisme s'affirme par l'intermédiaire de cellules nerveuses dont les ramifications protoplasmiques s'unissent en des réseaux nerveux qui traversent le tissu musculeux. Le rôle régularisant de ces réseaux périphériques profite du principe antagonistique: tonogenèse et tonolysie.

Or, la différence de leur innervation, surtout celle du seuil de l'excitant de l'inhibition centrale dans les muscles rapides du squelette et celle de la tonolysie périphérique (et centrale?) chez les muscles non-striés, explique la différence de leurs propriétés mécaniques, telles que leurs réactions dites plastiques le donnent à penser.

Cette récapitulation de faits acceptés généralement depuis longtemps (PAVLOW, BETHE, BIEDERMANN) était utile pour faire mieux comprendre l'antithèse de la théorie neuro-musculaire du tonus posée par JORDAN (et VON UEXKUELL 22, 23).

A l'égard des muscles creux lisses de quelques Invertébrés (Coelenthérates, Mollusques, Ascidies) JORDAN a distingué un troisième type d'innervation; il donne également une autre explication du comportement myogène pendant la réaction plastique de ces sacs musculeux. Il n'y constata pas l'intermédiaire d'impulsions pour la régularisation du tonus, ni que la tonogenèse ni la tonolysie dépendissent d'influx comme nous les connaissons du système nerveux des Vertébrés. La réaction plastique des muscles est attribuée à des propriétés mécaniques du tissu musculaire.

Quels étaient les phénomènes expérimentaux qui poussaient JORDAN à contredire les conceptions courantes? On va le voir ci-après.

D'abord: Quel que fût l'excitant utilisé, JORDAN n'a jamais observé une distension du muscle au sujet neuro-musculaire étudié (e. a. le pied de l'escargot). Il a constaté *soit* l'absence de réaction, *soit* une secousse qui se termine par un relâchement rapide et subit. Le muscle ne cède pas dans une mesure plus considérable à la force de traction exercée à nouveau par le poids de charge; il ne se rend pas non plus à une nouvelle tentative si auparavant il n'a toléré aucun allongement (7). Cependant la fonction tonolytique est prépondérante à la tonogenèse chez l'objet qui possède encore le ganglion suboesophagien où se trouve le centre du tonus pédal. On pourrait donc s'attendre à ce que les axones tonolytiques des nerfs pédieux — qui avant l'extirpation du ganglion relient celui-ci au réseau nerveux du pied — réagissent premièrement ou du moins d'une manière prépondérante à l'excitation.

En second lieu, l'application de la cocaïne en dose légère ne favorise pas la tonolysie quand on excite les nerfs pendant l'anesthésie progressive. Pourtant la cocaïne en elle-même cause une détente du muscle. Si des impulsions antagonistiques entraient en concurrence, la fonction accélérante devrait s'atténuer en premier lieu sous l'influence de la cocaïne. Dans ce cas, le système tonolytique devrait l'emporter progressivement et finalement s'affirmer seul, aussitôt que l'accélération serait totalement paralysée. JORDAN ne constata que des secousses ou même une absence totale de réaction.

Les résultats décrits ci-dessus démentaient donc les prévisions. De là, l'hypothèse tout à fait logique que, par rapport à la régularisation du tonus pédieux, *une activité des nerfs fondée sur la conduction d'influx antagonistiques serait incompatible avec le comportement musculaire constaté.* Au lieu d'un tel mécanisme, JORDAN supposait une influence réciproque entre le muscle et le système nerveux (le réseau nerveux et les ganglions superposés: le ganglion pédal et le cerveau) suivant le principe de l',,isostasie'' — cf. équipotentiel (7, 9). Le système neuro-pédal tendrait à maintenir une condition d'activité équivalente. Une hausse du travail ganglionnaire exciterait l'action tonique musculaire, une baisse la diminuerait. Réciproquement, l'activité périphérique se répercuterait proportionnellement à ,,une diminution de potentiel'' sur celle des centres supérieurs.

Au cas d'une réaction musculaire sous la forme d'une secousse, déclanchée par une excitation accommodante, le raccourcissement rapide est suivi par un relâchement instantané. *Après la contraction, le muscle ne se laisse allonger à nouveau qu'à la fin de la descente de la secousse.* Pendant le raccourcissement, la résistance tonique — présente auparavant — n'est donc pas rétablie, de sorte que le myomécanogramme enregistré monte rapidement au sommet aigu, puis il fait une chute brusque.

Le rallongement ne répète pas la réaction précédente d'allongement, mais il change après le relâchement brusque à angle vif, en faisant durer le comportement plastique. *Il est donc manifeste que le tonus n'est pas simplement une fonction de la longueur musculaire mais encore de la résistance. De ces contrastes* JORDAN *a conclu que cette résistance ne serait pas fondée sur la contraction, produit de la synthèse de secousses* des entités musculaires, comme nous la connaissons chez les Vertébrés et qui assure un raccourcissement soutenu, comparé au tétanos parfait du muscle tétanisé. L'insuffisance tonique ne serait pas le résultat d'un antagonisme entre la contraction et la décontraction commandées, avec une prépondérance de la dernière.

D'autre part, JORDAN a constaté *une correspondance remarquable entre les phénomènes toniques* des muscles lisses creux *et les phénomènes d'allongement* se manifestant dans les modèles plastiques. *Cette correspondance démontrerait l'identité des propriétés fondamentales* du modèle et de la masse musculeuse. L'état mécanique serait entretenu par des fonctions vitales, *la réaction plastique serait purement physique*, entièrement passive. JORDAN a attribué l'augmentation de la résistance pendant l'allongement à des „effets mécaniques comme la „consolidation'', en rapport avec la dimension du prolongement de la longueur et comme l'„accumulation'' en proportion avec la vitesse de l'allongement (8, pour l'explication voir 14, p. 431).

Toutefois, le physiologiste JORDAN a donné du tonus des muscles creux des Invertébrés une image bien intéressante et suggestive, d'ordre purement physique. Cette conception offrait en même temps la possibilité d'accepter un principe tonique qui joue comme un frein fixé sans la dépense d'un surplus d'énergie, par un métabolisme accru. Ce qui serait en concordance avec l'absence d'une augmentation mesurable du CO^2 rejeté, bien que la résistance tonique s'agrandisse considérablement.

Contentons-nous du résumé ci-dessus.

Les nouveaux résultats.

La très séduisante et curieuse théorie de JORDAN a suscité des critiques sérieuses e. a. de son maître BIEDERMANN. Les réussites expérimentales des quatres dernières années (1941—1945) nous permettent d'éliminer l'antithèse âgée de quarante ans. Enumérons les faits nouvellement acquis:

1o. *Nous avons réussi à déclancher une détente du pied de l'escargot au moyen de l'excitation* des ganglions supérieurs (G. cérébroïde et G. pédal 10, 9, 16) ainsi que des nerfs qui relient l'anneau oesophagien avec le réseau nerveux intramusculaire (13, 14, 15). Il suffit pour cela d'employer des stimulus faibles, de force au moins égale au seuil de la tonolysie. Une excitation plus forte ne déclanche qu'une contraction. Quand l'intensité de la stimulation est modérée, les deux réactions antagonistiques (contraction et décontraction) peuvent se compenser, de sorte

qu en apparence il n'y a pas d'ébranlement ni d'effet musculeux. Dans d'autres cas les réactions se succèdent: une secousse est suivie d'un hyper-relâchement ou autrement dit d'une hypotonie. Nos expériences démontrent l'exactitude de notre point de vue d'une manière plus convaincante que le dispositif expérimental de BIEDERMANN dans sa fameuse étude sur la physiologie comparée des mouvements péristaltiques (2, 3).

2o. Notre étude (10) de la fonction régulatrice du tonus des ganglions pédieux a révélé *que nous avons affaire à deux centres:* un centre *dorsal* qui provoque l'accroissement du tonus et un centre *ventral* qui en cause la diminution. Il est possible d'exalter et de calmer ces centres, autant par des moyens chimiques que par l'action électrotonique du courant continu.

Ces deux complexes de phénomènes éliminent d'une manière concluante l'objection contre le fonctionnement du système nerveux — ici dans le pied de l'escargot — par la conduction d'impulsions, soit tonolytiques, soit accélérantes. La contradiction entre l'effet anesthésique causé par l'application de la cocaïne du côté dorsal du ganglion sousoesophagien en lui-même (tonolysie) et la réaction sous la forme d'une contraction, réponse à l'excitation du système nerveux malgré l'anaesthésie progressive, peut s'expliquer comme suit: Par la séparation locale des deux centres toniques, le centre ventral autonome tonolytique échappe en premier lieu à l'émoussement, de là une prépondérance de l'inhibition. En outre l'échelonnement de l'intensité des excitants employés doit avoir été insuffisant, tandis que l'excitant électrique est facile à graduer.

Soumettons ensuite à la discussion la théorie musculaire physique de la réaction d'allongement plastique des sacs musculeux chez les Invertébrés. A l'égard de ce problème nous disposons des nouvelles données suivantes:

3o. *Souvent la résistance tonique* au lieu d'être constante *montre pendant l'allongement un acroissement progressif.* Nous avons trouvé que *cette augmentation n'est pas un phénomène purement physique,* fondé sur les propriétés mécaniques du muscle. *C'est une réaction neuro-musculaire active,* ce qui est prouvé par l'antagonisme entre l'accroissement du tonus causé par l'allongement et l'excitation tonolytique. L'influence répressive de celle-ci est moins effective sur la tonogenèse, quand l'excitation est appliquée pendant la traction, que quand l'on provoque la répression avant que l'allongement se fasse sentir (14, 15).

4o. Nous constatons non seulement *un réflexe d'allongement accélérant* mais nous remarquons également que *la tension est perçue et même d'une manière spécifique: la tension faible* produite par une charge d'étirement plus légère *cause une tonogenèse;* une charge plus lourde qui s'exprime par *une tension forte fait diminuer le tonus* (11, 12, 14).

Comment expliquer le mécanisme de ces réflexes? Raisonnons d'abord, nos conclusions seront ensuite soumises au contrôle de l'expérience. La réussite de l'excitation tonolytique ouvre l'alternative: interaction au moyen d'influx antagonistiques ou, sans impulsions, obéissant à la loi de l'isostasie. Au début, l'excitant était seulement appliqué au ganglion ou aux nerfs extra-musculaires. De là l'effet modérateur ne témoigne en faveur du rôle d'impulsions concurrentes que par rapport à la régularisation centrale du tonus. Le trio de réflexes se produit cependant après que le pied est privé des ganglions supérieurs, dans la périphérie ainsi que dans la préparation neuro-musculaire centrale. La question se pose donc de savoir si, dans la périphérie, il s'agit également d'influx antagonistiques. Si l'expérience ordinaire montre que, en ce qui concerne la périphérie, les choses se passent suivant les rapports qui existent dans le système central où tout argument contre le jeu d'impulsions antagonistiques est éliminé, la même théorie serait acceptable pour le pied privé du système nerveux extramusculaire. En collaboration avec M. DE JONG nous avons approché ce problème en étudiant la loi qui relie l'excitabilité du système neuro-myo-épithélial pédieux au caractère de l'effet déclanché. Voici le résumé de l'étude:

5o. DE JONG (6) a déterminé *l'influence* de la violence de l'excitant au moyen *d'un courant inducteur appliqué au muscle pédieux*. La sole ventrale du pied reposant sur l'une des électrodes et l'autre électrode touchant les flancs pédieux du côté dorsal, les courants induits traversent tout le tissu musculaire. De cette manière DE JONG *a trouvé une relation identique à celle que nous-même avons constatée* en stimulant le ganglion pédal ou les nerfs pédieux: *tonolysie avec des stimulus tout juste liminaires, hypertonie par excitation plus intense*. Entre ces intensités de stimulation faible et forte, on trouve des excitants moyens qui restaurent le tonus (raccourcissement *et* résistance tonique). En effet, si nous stimulons la préparation centrale et la préparation périphérique au moyen d'intensités progressives, il s'agit ici d'une concordance de l'excitabilité successive des systèmes de freinage et d'accélération des unités musculaires. On se trouve donc autorisé à conclure que le fonctionnement du système neuro-musculaire dans le pied de l'escargot peut être formulé d'après la théorie courante des réflexes. De plus, les trois réflexes décrits ci-dessus (celui de traction et ceux de tension: le réflexe inhibitif et celui de tonogenèse) nous mettent en état d'expliquer non seulement la réaction plastique musculaire et son analogie avec le comportement du modèle inanimé, mais encore les différents types de contraction qui résultent de l'excitation du pied, dont la secousse (sans restauration de la résistance) et la contraction dite tonique (rétablissant la résistance) représentent deux formes différentes.

Nouvelles conceptions.

1o. Nous appuyant sur les constatations précédentes, nous allons donner l'explication requise sur la nature de la *réaction d'allongement*.

a. Il est maintenant possible de comprendre pourquoi l'on peut s'attendre à la production d'une réaction à résistance constante: quand le réflexe d'allongement est suffisamment contrarié par le réflexe tonolytique de tension forte, la production du tonus est éliminée par la tonolysie. Cette combinaison d'éléments antagonistiques a été mise en évidence par JORDAN comme réaction d'allongement „idéale". De là, sa théorie que le tonus serait de nature physique sans dépense d'énergie.

b. Au fur et à mesure que, pendant l'étirement, la dimension de l'accroissement de la longueur musculaire — donc l'amplitude de l'allongement — s'agrandit, et en proportion avec la durée de celui-ci, la production de résistance est plus intense. Dans le modèle plastique (de la gélatine, du caoutchouc) la résistance augmente d'une manière analogue par la „consolidation".

c. Plus l'allongement s'opère vite, plus le réflexe annexe se fera sentir; dans le modèle, la résistance s'accroît de la même manière par l'„accumulation".

Il existe donc un rapport déterminé entre la mesure et la vitesse de l'allongement d'une part et le progrès de la résistance tonique de l'autre. *Nous rencontrons ici alors à base physiologique une imitation magistrale de la réaction de l'allongement qui dans l'expérience modèle est fondée sur les propriétés mécaniques de la substance plastiquement extensible.*

Un allongement rapide produit par un poids d'étirement plus lourd provoque donc une contre-réaction réflexe qui protège le muscle contre une détente excessive. Dans le pied en relation avec le ganglion sous-oesophagien, le jeu autonome du centre tonolytique favorise un allongement vif et la contre-réaction est intensifiée par le centre dorsal. La conséquence en est que la préparation centrale fait généralement cesser déjà la distension du pied alors que le muscle isolé cède encore à la traction, bien que retardée, en comparaison avec son allongement primitif.

2o. Reste à savoir comment la multiformité du raccourcissement et du rallongement du pied de l'escargot est déterminée, autrement dit d'où proviennent *les différents types de contraction*?

JORDAN a toujours mis la contraction dite tonique — où au sommet du raccourcissement la charge d'étirement subit une nouvelle résistance — en opposition avec la secousse qui, le sommet dépassé, change en une distension rapide et dont le mécanogramme se superpose à celui de l'allongement ainsi qu'à l'ascente et la descente de la contraction tonique (19).

a. Après notre analyse des réflexes, il est admissible que *la contraction tonique* réussisse mieux si la restauration de la résistance n'est pas contrariée par des impressions tonolytiques qui s'affirment:

α. comme endurcissement d'un étirement antérieur causé par une charge plus lourde; et/ou

β. accélérées par un raccourcissement qui doit vaincre une résistance externe non négligeable (le chargement du muscle, la friction dans l'appareil d'étirement) de sorte que la contraction doit être accompagnée d'un développement de tension plus forte et ce qui suscitera le réflexe usuel; et/ou

γ. déclanchées par l'emploi d'excitants liminaires.

La secousse sera favorisée quand la diminution de la résistance pourra jouer un rôle important à côté de la contraction brusque, aussi bien par la réduction de l'assourdissement qui contrarie la mobilité — ce qui se montrera dans le mécanogramme sous forme d'une ascente rapide — que par l'accélération du relâchement. Cette dernière cause un rallongement rapide du muscle, ce qui produira de nouveau de la résistance tonique. A l'égard de cette production nous rememorons, quoique bien convaincu que le muscle nous offre tous les intermédiaires entre les deux types extrêmes:

α. Si cette résistance s'exécute en proportion idéale, la descente de la secousse se termine à angle vif au niveau où l'allongement précédent avait été interrompu;

β. Une prépondérance de la tonolysie produira une contraction suivie d'une hyperdétente. JORDAN la formulait en ces termes: „La secousse casse le tonus".

γ. Un excès de production de la résistance peut provoquer des mouvements contracturiformes, c'est-à-d. dont le mécanogramme présente une concavité en dessous. JORDAN a toujours disqualifié ce dernier type de contraction comme pathologique de sorte qu'il lui déniait toute importance physiologique.

La plasticité de *tous* les muscles est donc basée sur le même principe, qu'elle se rapporte aux muscles striés somatiques ou aux muscles non striés splanchniques des Vertébrés ou aux muscles lisses des Invertébrés. à savoir: l'antagonisme de l'accélération et du freinage, et celui de la contraction et de la décontraction qui lui est conforme. Faut-il en conclure que le mécanisme interne est entièrement uniforme? Il serait intéressant de le savoir. Mais actuellement aucune précision n'est encore permise à cet égard. Ce qu'il importe de ne pas oublier, c'est qu'il faut distinguer le processus neuro-physiologique du fonctionnement myo-physiologique.

Par rapport au premier la question essentielle se pose: „Quelle est la nature du commandement transmis par la dernière cellule nerveuse à l'entité musculaire?" Dans le système somatique il n'existe que des impulsions centrifuges qui déclanchent des contractions. Dans le réseau nerveux des sacs musculeux, il est possible que des influx antogonistiques soient émis par la cellule nerveuse périphérique vers l'unité motrice musculaire. Ou bien il ne s'agit que d'impulsions accélérantes, de sorte que l'inhibition s'exé-

cute toujours intraneuralement. A vrai dire, s'il était ainsi, on serait autorisé à croire que le fonctionnement neuro-musculaire ne présente qu'un principe monistique: l'inhibition centrale entre des influx centripètes antagonistiques déterminant les impulsions centrifuges qui ne produisent que plus ou moins de contractions. Dans ce cas, les contrastes sont dûs à des différences anatomiques: les dernières cellules nerveuses du système somatique des Vertébrés se trouvent dans la colonne rachidienne, celles du système viscéral dans le réseau périphérique.

Cependant un tel monisme ne peut pas être accepté „à priori". En premier lieu il existe la différence que le muscle squelettique se distend spontanément, tandis que le muscle lisse creux — au moins pour un relâchement rapide — exige des influx qui déclanchent la décontraction. En second lieu, l'étude de la contraction des fils d'actomyosine tendus mécaniquement (SZENT GYORGYI) et celle de la contractilité des fibres musculaires striées isolées, dont les éléments contractiles sont soumis à l'extension par un champ de force électrique (BUCHTHAL), montre deux types de fonctionnement exactement parallèles à la différence rememorée entre le muscle lisse et le muscle strié: Le fil d'actomyosine exige en plus d'un chimostimulant (ions de potassium) pour la contraction (20), un stimulant similaire (ions de magnésie) pour la décontraction (21). Au contraire la fibre striée se relâche immédiatement par la restauration spontanée du champ électrique dont l'affaiblissement a produit la précédente contraction. Mais il faut à ce déchargement partiel un stimulant nerveux (4). — Le parallélisme nous amène à préférer la conception dualistique concernant le fonctionnement du système neuro-musculaire. A notre avis, le mécanisme du muscle lisse représente l'état primitif dans lequel la faculté de soutenir le raccourcissement sans dépense d'énergie est développée au détriment de la mobilité rapide. Nous estimons que le muscle strié s'est spécialisé en faveur du mouvement rapide; il en résulte que le tonus est fondé principalement sur le tétanos qui exige une augmentation considérable du métabolisme. Nous croyons cependant reconnaître quelques vestiges du mécanisme tonique primitif dans le rôle du système autonome en rapport avec le muscle strié: des effets positifs sous l'influence de la chaîne sympathique (VAN DIJK, 5) et des effets négatifs produits par le système parasympathique (BARISCHNIKOW, 1). Ces effets ne sont pas seulement de nature trophique (par l'intermédiaire de la circulation, etc.), mais également de nature motrice (au moyen d'éléments toniques du tissu musculaire).

Résumé.

Une étude approfondie des réflexes du pied de l'escargot et des lois de son excitabilité nous a appris que dans tout le système neuro-musculaire, un principe antagonistique s'affirme entre la contraction et la décontraction (le jeu musculaire), déclanchées par des influx accélérants et des im-

pulsions frénatrices (le rôle des nerfs). Ce principe est en vigueur à la fois à la périphérie (le muscle et son réseau nerveux) et dans le système nerveux extramusculaire (les nerfs pédieux et le ganglion sous-oesophagien).

Il existe trois réflexes toniques, à savoir:

1o. celui de l'allongement;

2o. ceux de tension: *a.* l'un de tension faible;
b. l'autre de tension forte.

Les deux premiers font naître des influx accélérants, le troisième produit des impulsions tonolytiques.

Ces résultats nous permettent d'expliquer d'une part la réaction plastique du muscle et son analogie avec la réaction d'allongement du modèle inanimé et d'autre part, les différents types de contraction. La conséquence en est que les théories de JORDAN en ce qui concerne la nature physique de la réaction d'allongement tonique du pied de l'escargot *en entier* (le jeu musculaire) et le principe de l'isostasie (le rôle nerveux) dans la régularisation du tonus n'ont gardé qu'une valeur historique. Il faudra que de nouvelles expériences sur la fibre isolée du muscle lisse, parallèles à celles de BUCHTHAL, nous procurent l'analyse de son comportement physiologique et mécanique et nous donnent des éclaircissements — en ce qui concerne la fibre *isolée* — sur la validité éventuelle de la théorie du tonus musculaire de JORDAN.

Enfin, du point de vue de la physiologie comparée, nous avons adopté la conception dualistique concernant le fonctionnement du système neuromusculaire: A notre avis, le mécanisme du muscle lisse, exigeant des impulsions frénatrices efférentes, représente l'état primitif, dans lequel la faculté de soutenir le raccourcissement sans dépense d'énergie est la fonction capitale. Nous admettons que le muscle strié qui se relâche spontanément, se soit spécialisé en faveur du mouvement rapide.

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Medicine. — *About arenicochrome and its possible significance as a mesocatalyst.* By G. O. E. LIGNAC. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of October 27, 1945.)

By arenicochrome I mean the green pigment which is found in the arenicola marina (lugworm, fig. 1 and 2) in a corpuscular form. The green, more or less globular granules are found not only in the single layered, pseudo-stratified, columnar epithelial cells, but also scattered in the sub-epithelial loose connective tissue (coloured plate, fig. 1 210 × and fig. 2 675 × enlargement). These little pigment-balls have a diameter which varies from 0,25 to 1,30 μ (to the nearest 0,01 μ) [1]. In the loose connective tissue among the green pigment one meets with a second pigment that differs from the arenicochrome by its dark-yellow, yellow-brown to dark-brown colour, in that it mostly occurs in groups and by its coarser. mostly angular granules. This latter pigment, which, morphologically and histio-chemically speaking, can be put on a level with melanin, is also found between the outer and inner muscle-tunic, especially arranged round the blood-vessels, thus also between the bundles of the inner muscle-tunic and heaped up in very great quantities along the greater blood-vessels, which are bounded by the inner muscle-tunic on the proximal side. The third pigment is the blood-pigment, the erythrocruorin, which in the arenicola occurs dissolved in the blood-plasm (cf. coloured plate).

In 1899 PIERRE FAUVEL [2]) wrote a study about the above-mentioned two pigments. In the treatise he calls our arenicochrome "lipochrome jaune" and states the solubility of this pigment e.g. in alcohol. If this solution is exposed to light, it becomes brown and gives a fine, black precipitation, which cannot be dissolved in water or ammonia. If a few drops of ammonia are added to the alcoholic solution the colour becomes intensely green and even in light the formation of the black precipitation is prevented. On the other hand, addition of a few drops of hydrochloric acid or other acids to the alcoholic solution of the "lipochrome jaune" accelerates the formation of the brown, brown-black precipitation. These precipitations cannot be dissolved in water, alcohol or ammonia.

FAUVEL tries, among other things by tissue-sections, to make it plausible that the brown pigment is a chemical modification of the "lipochrome

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1) A very great number of measurements by means of the measuring-clock of S. T. BOK have been made by my ex-assistant J. F. WALBURGH SCHMIDT. The coloured plate was also made by him. In literature I found no picture of this pigment and therefore it is added here.

2) PIERRE FAUVEL, Sur le pigment des Arénicoles. Comptes rendus des séances de l'Académie des sciences. Bd. **129**, 1273 (1899).

G. O. E. Lignac: *About arenicochrome and its possible significance as a mesocatalyst.*

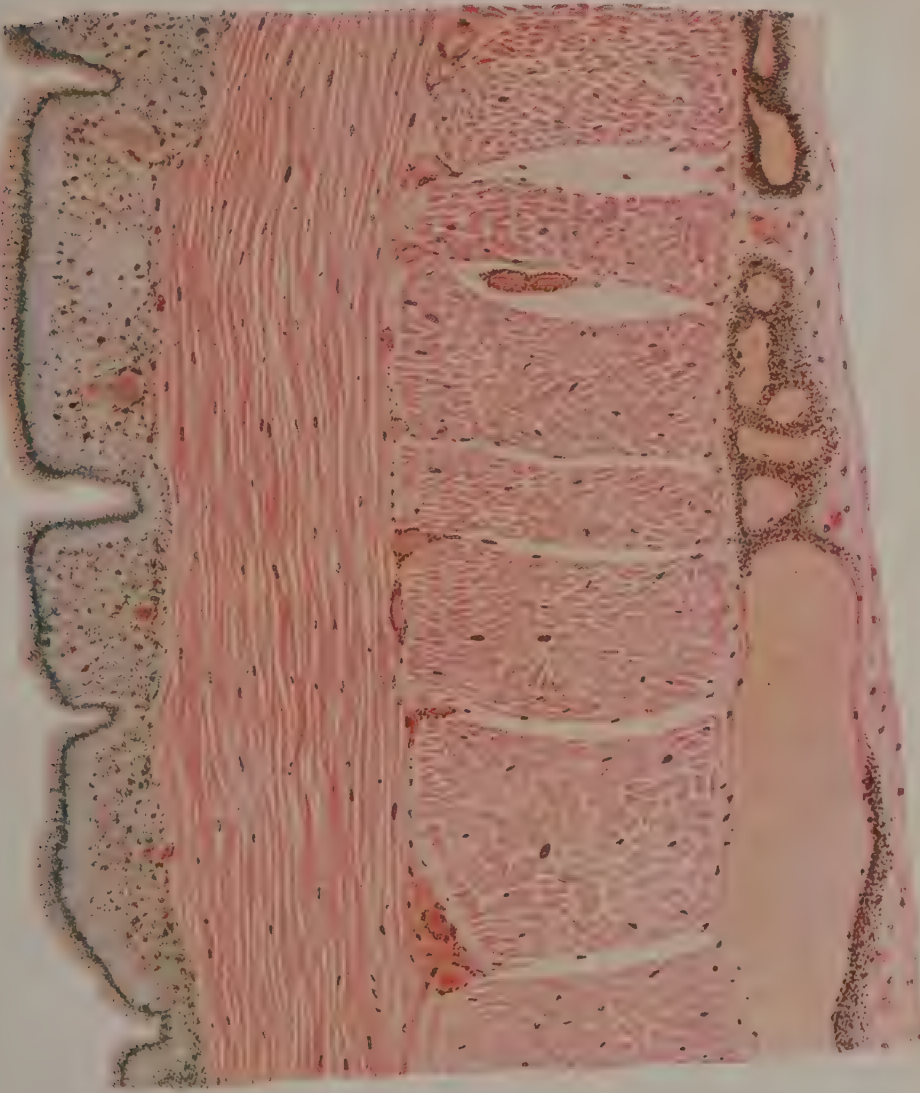

Figure 1. Section of the skin and muscle-tunics of the arenicola marina (enlargement 210 ×).

Figure 2.
The single layered, pseudo-stratified, columnar epithelial cells with the arenicochrome-pigment (enlargement 675 ×).

G. O. E. LIGNAC: *About arenicochrome and its possible significance as a mesocatalyst.*

Arenicola marina, seen from the ventral side.

Arenicola marina, seen from aside.

jaune". Acids further the coming into being of this modification. To sum up it may be said that FAUVEL has shown that the arenicochrome can be dissolved in alcohol, is perpetuated by alkali, is changed under the influence of light, especially when acids are present, into a brown, brown-black pigment. The chemical structure of the "lipochrome jaune" would show small differences in various kinds of arenicola.

C. FR. W. KRUKENBERG [1]) classes the above-mentioned pigment among the uranidins. In how far the pigments of very different origin according to their chemical properties and chemical structure may be united into one large group and in how far the arenicochrome mentioned by me may be considered as the prototype of this group of pigments, must undoubtedly be further determined by a thorough chemical examination.

This enumeration gives rise to the supposition that the transition of the green pigment to the brown-black precipitation is brought about by oxygen. I have shown this in a simple manner. If one makes frozen sections of the arenicola and then at room-temperature exposes them to the action of 3 % H_2O_2, it is seen within 24 hours that the pigment-granules of the arenicochrome have changed from green into brown, brown-black. In how far can we assume now that the arenicochrome has changed into "melanin"? Without going further into the matter of the melanins [2]), it may be advanced here that histiochemically melanins can be identified by 2 reactions and distinguished from other pigments, namely melanins are decolourized by H_2O_2 and in the dark they reduce a 1 % $AgNO_3$ solution. It has become clear to me that in the first place the dark-yellow, yellow-brown to dark-brown pigment of the arenicola promptly shows these two reactions and so can be classed among the melanins also histiochemically; secondly that the arenicochrome, which has been oxidized brown-brown-black by H_2O_2 also shows the two above-mentioned reactions. So as far as is possible at present, it has been shown by me that by oxidation the arenicochrome can change into a melanin. So in the arenicochrome one meets with a *coloured* pre-melanin, which occurs more or less durably in the *living* organism of the lugworm. Like the melanin the arenicochrome reduces in the dark both the 1 % watery solution and the 1 % alcoholic $AgNO_3$ solution. All the above-mentioned reactions have been carried out on frozen sections of not only worms fixed in alcohol, but also on those of fresh ones.

What biological signification should be ascribed to this coloured pre-melanin, which occurs in the lower organism and is apparently more or less durable? Before trying to answer this question it is first necessary to examine further the chemism of a melanin known to us and occurring

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[1]) C. FR. W. KRUKENBERG. Vergleichend-physiologische Vorträge Bd. I. 1886.

[2]) Read for this: Melanins, their chemistry and their significance by my ex-assistant, the biochemist Dr W. L. C. VEER. Chemisch Weekblad, Deel 37, No. 16 (1940).

in man. The investigation by the English biochemist H. S. RAPER [1]) (Manchester) has appeared to be of fundamental significance for the knowledge of the chemism of the melanogenesis. According to RAPER tyrosinase, when oxygen is present, causes an OH group to arise on the orthoplace in the benzol-chain of the tyrosin. Then an orthodioxybenzol has come into being, called dopa (BR. BLOCH, 1917), which is highly autoxidisable and is oxidized to the corresponding quinone. From this indol forms itself by ring-closure and back-formation of the dioxybenzol (intramolecular shifting of atoms). By oxidation this product changes into the corresponding *quinone*, which is coloured *red* and was called "red substance" by RAPER. For the rest it became evident to RAPER and his associates that by an intramolecular shifting of atoms and after this by simultaneous decarboxylation this "red substance" changes again into colourless dihydroxyindoles, which are highly autoxidisable. These latter intermediate products quickly change into melanin by oxidation. In my laboratory the "red substance" appeared to be an intractable substance. This red quinone, if obtained by tyrosinase from a tyrosin-solution, can*not* be shaken out, it is again decolourized very easily and then returns to the dioxy-indol-compound, e.g. by ascorbic acid, or if one tries to isolate it in vacuo or in an indifferent gas; when air or oxygen (autoxidisable) is present, the substance is quickly coloured black (formation of melanin). W. L. C. VEER [2]) has succeeded in isolating the "red substance" with phenylhydrazin as a monophenylhydrazone (1939).

Yet it appears that this intractable substance can occur more or less durably in the lower organism. MAZZA ond STOLFI [3]) succeeded in isolating the red substance — also as a phenylhydrazone — from the worm *Halla parthenopaea Costa* and in identifying it as the "red substance" of RAPER. This pigment of the worm, which may reach a length of 50—80 cm and is coloured orange and red on the back, lives in the Mediterranean (Naples and Genoa) and the Atlantic Ocean (Cadix), is called hallachrome. When the worm dies, it becomes brown and a black juice comes from it. The question arises how the "red substance" is more or less perpetuated in the *living* organism of this worm? I found nothing in literature about the histiology and the histiochemic reactions of this pigment.

It follows from the conduct of the "red substance" that it forms a phase in a reversible redox-system. MAZZA and STOLFI (l. c.) suspected already that the hallachrome in the worm has the function of a reversible hydrogen

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[1]) H. S. RAPER and A. WORMALL, Biochem. J. **17**, 454 (1923); **19**, 84 (1925); H. S. RAPER, Biochem. J. **20**, 735 (1926); **21**, 89 (1927); Fermentforschung **9**, 206 (1927); Physiol. Rev. **8**, 245 (1928); Ergebnisse Enzymforschung **1**, 270 (1932).

[2]) W. L. C. VEER, Recueil des travaux chimiques des Pays-Bas. T. **58**, 9/10 (septembre et octobre 1939).

[3]) F. P. MAZZA and G. STOLFI, Arch. sci. biol. **16**, 183 (1931).

transmitter. FRIEDHEIM [1]) stated that this quinone augments the respiration of erythrocytes many times. This pigment can be reduced reversibly to its leuco-compound with cystein or with hydrogen and colloidal Pd. Possibly the "red substance" also plays the part of respiration-ferment in man (W. L. C. VEER l. c.).

SZENT GYÖRGYI [2]) has pointed to the significance of ortho-quinones when there is biological oxidation. The catalysis via quinoid pigments plays a considerable part in the metabolism of microbes, e.g. the pyocyanin in the respiration of the bac. pyocyaneus, possibly also the phthiocol (a naphtoquinone-derivative) of the bac. tuberculosis. One might speak here of a possible respiration via the pigments. Naphtoquinones are often found in phanerogamen as reversible redox-systems. Quinone-catalysis would play an important part in the metabolism proper and a type of respiration seems to be meant where beside autoxidisable polyphenols peroxydases also contribute; further the ascorbic acid also takes part in these processes (SZENT GYÖRGYI c.s.).

According to CARL OPPPENHEIMER [3]) *meso*catalysts are catalysts which are not enzymes, but shove themselves in among the enzymatically catalyzed processes; they are what the English school calls "carrier" and THUNBERG "hydrogen-transmitters". Specific mesocatalysts are chemical substances which do not arise from the stages of normal destruction of cellular processes, but are formed for a certain function. Up to now two groups have been known: *the quinoid pigments,* respectively chromogens of various kinds and non-chromogenic substances such as glutathione and ascorbic acid. In connection with the question as to the biological significance of the arenicochrome we are here especially interested in the quinoid pigments. Those known at present are reversible redox-systems, which absorb hydrogen from the donator-dehydrasis-combinations with a lower potential (mostly only one atom: semiquinone-formation) and cede to an acceptor with a higher potential, so keep the hydrokinesis going. When the mesocatalysts are autoxidisable they can also hydrate oxygen directly, i. e. perform a respiration (OPPENHEIMER l. c.) Very little is known as yet about the details of their action.

The "red substance" or the hallachrome and the arenicochrome are both coloured pre-melanins, occurring in lower organisms. They are both autoxidisable, the arenicochrome especially in acid medium. In the *living* organism they appear to be more or less durable. By oxidation both pigments change into a melanin. The pigments differ in colour, the hallachrome does not reduce an $AgNO_3$ solution, the arenicochrome does; the arenicochrome is found in a corpuscular form in the organism of the

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1) E. A. H. FRIEDHEIM, Compt. rend. soc. biol. **111**, 505 (1932); Naturwissenschaften **21**, 177 (1933); Biochem. Z. **259**, 257 (1933).

2) A. SZENT GYÖRGYI, Z. physiol. Chem. **254**, 147 (1938).

3) CARL OPPENHEIMER, Chemisch Weekblad **37**, 259 (1940) and **37**, 282 (1940).

lugworm, I have not been able to find anything in literature about the state of aggregation of the hallachrome in the worm. The chemism of the hallachrome is known to us, thanks to the investigations by RAPER. The chemical structure of the arenicochrome is not known to us. The only thing we are certain about is that the arenicochrome is an organic pigment and is easily oxidized to melanin. So there is a superficial likeness. However, the question remains justified — though we must remain aware of the fact that we argue here by analogy — whether the arenicochrome is perhaps one of the quinoid pigments and derives from this its biological significance as a mesocatalyst, i. e. can bring about an "extra-respiration" (pigment-respiration) and, according to circumstances, can accelerate the end of an existing respiration.

Attempts at isolating the arenicochrome in a pure state have up to now failed in my laboratory (W. L. C. VEER). The arenicochrome dissolves with a yellow-brown colour in alcohol, light green in 5 % and 15 % watery NH_4OH solution, light yellow in 4 % watery NaOH solution, light green in 50 % pyridin and yellow-brown in $(NH_4)_2S$. VEER could also state, on the impure arenicochrome that with a diluted H_2O_2 solution the green colour first changed into a brown-black one and disappeared after this (cf. histiochemic reactions). Concentrated H_2O_2 (30 %) causes almost direct decolourizing of the arenicochrome.

It need hardly be said that in order to answer definitely the above-mentioned questions one should have the arenicochrome at one's disposal in a pure state.

So to sum up the following can be stated:

The arenicola marina harbours two pigments:

1. Green, globular corpuscula (diameter varying from 0,25 to 1,30 μ); it is suggested by me to call this "lipochrome jaune" of PIERRE FAUVEL *arenicochrome*.

2. Coarser, angular granules of a dark-yellow to dark-brown colour, which histiochemically show the characteristics of *melanin*.

It appeared to me histiochemically that the arenicochrome is autoxidisable in acid medium, is oxidized by oxygen to the above-mentioned melanin-pigment. So the arenicochrome is a *coloured* pre-melanin, which occurs more or less durably in the *living* organism of the lugworm.

Analogous to the hallachrome ("red substance" of RAPER) the following hypothesis is proposed by me about the biological significance of the arenicochrome: the arenicochrome derives its significance as a meso-catalyst from the fact that the arenicochrome is probably a quinod pigment.

Medicine. — *L'action inhibitrice des métaux sur la croissance du B. tuberculeux.* III. *Germanium, étain et plomb.* By ONG SIAN GWAN. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of October 27, 1945.)

1. Nous avons montré dans des mémoires précédents [1]) l'action inhibitrice d'arsénic, d'antimoine et de bismuth métalliques sur la croissance du B. tuberculeux. Il est important de savoir si le germanium qui, dans le tableau périodique est situé à côté de l'arsénic, n'a pas également une action inhibitrice sur la croissance du B. tuberculeux.

D'après ISHIWARA [2]) le germanium montre une action inhibitrice sur le cancer de souris. HAMMET, NOWREY et MÜLLER [3]) ont montré que l'injection de GeO_2 chez des rats neufs fait augmenter le nombre de globules rouges. KAST, CROLL et SCHMITZ [4]) ont utilisé le GeO_2 dans le traitement de l'anémie. Ils ont noté une augmentation passagère de globules rouges. Par contre, le GeO_2 n'a aucune action dans l'anémie pernicieuse [5]).

Plusieurs auteurs ont montré que les composés de germanium ne sont pas très toxiques pour les animaux de laboratoire [6]). Dans des essais sur des souris nous avons noté que la souris blanche peut supporter une injection souscutanée de 10 mg de GeO_2.

2. *Action inhibitrice de Ge, Sn et Pb sur la croissance du B. tuberculeux.*

Les expériences ont été effectuées d'après la technique adoptée dans les mémoires précédents [1]). On utilise pour chaque métal et le témoin quatre cultures, pourqu'on puisse comparer les poids moyens de microbes obtenus. La différence entre les poids moyens est significante, si la probabilité P pourque, t dépasse $\pm\, t$ est égale ou inférieure à 5 p. 100.

Les métaux utilisés dans ces expériences sont: **1**. Germanium met. Merck. **2**. Germanium Hilger, contenant 99,995 p. 100 de Ge, 0.004 p. 100 de Ca, 0.001 p. 100 de Pb, très peu de C et de petites traces de S, As, Na et Ag. **3**. Sn in bac. KAHLBAUM, pro analyse. **4**. Pb HILGER con-

1) ONG SIAN GWAN, Verslagen Ned. Akad. v. Wetensch., Amsterdam, **53**, 345 et 353 (1944).

2) F. ISHIWARA, Ber. ges. Physiol., **49**, 615 et 177 (1929).

3) F. S. HAMMETT, J. E. NOWREY et J. H. MÜLLER, J. Exp. Med., **35**, 173 (1922).

4) L. KAST, H. M. CROLL et H. W. SCHMITZ, Proc. Soc. Exp. Biol., N.Y. **19**, 398, (1922)

5) M. E. ALEXANDER, Amer. J. Med., Sci., **166**, 256 (1923).

6) F. S. HAMMETT, J. H. MÜLLER et J. E. NOWREY, J. Pharmacol., **19**, 337 (1922). I. KEESER, Arch. f. exp. Path. u. Pharmakol., **113**, 232 (1926)

tenant 99,999 p. 100 de Pb, 0,0001 p. 100 de Bi et de petites traces de Cu et de Sb.

TABLEAU 1.

Action inhibitrice de germanium, d'étain et de plomb sur la croissance du B. tuberculeux.

Métal	Poids en mg				Poids moyen en mg	*t*	*P*
			A. 5 mg de métal/100 cc				
Témoins	780.5	844.0	468.0	(57.5)	697.50		
Ge	42.7	17.5	21.5	22.0	25.93	6.890	0.001
Sn	54.0	50.0	32.0	39.0	43.75	6.711	0.001
Pb	759.0	742.0	(65.0)	(77.5)	750.50	0.353	0.7
			B. 5 mg de métal/100 cc				
Témoins	818.0	803.0	822.0	829.0	818.00		
Ge	49.0	26.5	80.5	41.5	49.38	60.828	<<0.001
Sn	810.0	837.0	789.0	725.0	790.25	1.133	0.3
Pb	794.0	804.5	820.0	810.0	807.13	1.370	0.2
			C. 10 mg de métal/100 cc				
Témoins	782.8	793.5	821.0	803.5	800.20		
Ge	9.5	46.8	63.0	14.3	33.40	51.475	<<0.001
Sn	779.5	786.0	95.8	19.0	420.08	1.809	0.1
Pb	780.0	764.0	773.0	778.0	773.75	2.982	0.03
			D. 20 mg de métal/100 cc				
Témoins	92.5	474.0	212.0	282.5	265.25		
Ge	10.5	6.0	1.5	10.5	7.13	3.231	0.02
Sn	2.5	6.5	7.5	7.0	5.88	3.247	0.02
Pb	637.0	102.0	544.0	580.0	465.75	1.369	0.2
Témoins		(voir B)			818.00		
Pb	798.0	825.8	816.8	830.8	817.85	0.165	0.9
			E. 20 mg de métal/100 cc				
Témoins	670.0	639.0	552.5	627.5	622.25		
Ge Hilger	28.5	23.3	26.0	19.0	24.20	23.917	<<0.001
Sn	12.0	11.8	7.5	478.0	127.33	4.195	0.01
Pb	417.0	532.0	648.0	562.0	539.75	1.532	0.2

Les résultats obtenus sont présentés dan le tableau 1.

Groupe A. 5 mg de métal par 100 cc de milieu de culture. On ensemence avec la souche bovine Vallée, qui a subi 13 passages sur le milieu synthétique de Sauton. La culture utilisée est de 25 jours. Au cours de l'expérience la voile de B. tuberculeux d'une culture témoin et de deux cultures contenant le plomb est tombée, de sorte que les poids placés entre parenthèses sont trouvés trop petits. La durée de l'expérience est de

32 jours. Si l'on compare le résultat obtenu avec le témoin on constate que le germanium et l'étain ont donné une différence réelle.

Groupe B. On fait la même expérience que la groupe *A*. On ensemence avec la souche Vallée (culture de 11 jours) après le quinzième passage sur milieu de Sauton, (Vallée$_{15}$). La durée de l'expérience est de 28 jours. Comme dans l'expérience précédente le germanium a donné une différence réelle si l'on compare avec le témoin. Par contre, l'étain ne donne pas une différence réelle.

On ensemence en même temps avec ces cultures quatre flacons contenant 20 mg de Pb par 100 cc (voir groupe *D*).

Groupe C. 10 mg de métal par 100 cc. On ensemence avec la souche Vallée$_{16}$ d'une culture de 13 jours. La durée de l'expérience est de 31 jours. Le résultat obtenu est comparable à celui du groupe *B*. L'écart des poids vis-à-vis du poids moyen est dans ce groupe assez grand pour Sn; pour cette raison on détermine également les poids moyens et l'écart type à partir des logarithmes des poids obtenus. On obtient $t = 1{,}636$ et $P = 0{,}2$ ce qui correspond avec le résultat obtenu par la méthode courante (voir tableau 1).

Groupe D. 20 mg de métal par 100 cc. On ensemence avec la souche Vallée$_{9}$ d'une culture de 29 jours. Durée d'expérience 27 jours. Ge et Sn donnent une différence réelle en comparaison avec le témoin. Puisque dans le groupe C avec 10 mg de Pb la différence est significante, on fait dans ce groupe deux séries parallèles avec 20 mg de Pb afin de savoir si en augmentant la quantité de Pb la différence demeure réelle. Le résultat montre le contraire, il n'existe pas de différence entre le Pb et le témoin.

Groupe E. 20 mg de métal par 100 cc. On ensemence avec la souche Vallée$_{18}$ d'une culture de 14 jours. Durée d'expérience 26 jours. Cette expérience est une reproduction de la précédente. On utilise seulement le germanium très pur de Hilger afin de savoir si le résultat obtenu avec le germanium dans les expériences précédentes n'est pas dû à la présence d'impuretés dans le métal. Le résultat montre qu'il n'en est pas ainsi, puisque les résultats obtenus avec Ge, Sn et Pb sont comparables avec ceux de l'expérience précédente. Comme dans la groupe *C* on constate que l'écart des poids obtenus vis-à-vis du poids moyen est grand dans les essais avec Sn. Le calcul à l'aide des poids moyens et de l'écart type obtenus à partir des logarithmes des poids obtenus donne $t = 6{,}593$ et $P < 0{,}001$. La différence est maintenant très significante.

En résumé, le germanium donne dans toutes les expériences une différence de poids réelle et nette en comparaison avec le témoin. Dans le cas de Sn, on ne trouve pas dans les groupes *B* et *C* avec 5 et 10 mg de Sn une différence de poids réelle en comparaison avec le témoin. Par contre, dans les groupes A, D et E cette différence est réelle. Le plomb ne montre pratiquement aucune action inhibitrice sur la croissance du B. tuberculeux.

3. *Influence de la quantité de métal ajouté sur la croissance du B. tuberculeux.*

On peut comparer les résultats obtenus dans les différentes groupes d'expériences si l'on rapporte les poids obtenus à celui du poids moyen témoin et en supposant ceci égal à 1. En faisant ainsi on voit dans le tableau 2 que l'action inhibitrice de germanium augmente en même temps avec la quantité de métal ajouté. L'étain donne probablement le même résultat.

TABLEAU 2.
Influence de la concentration des métaux sur la croissance du B. tuberculeux.

Numéro d'experience	Quantité de métal mg/100 cc	Germanium	Etain	Plomb
Témoins	—	1.000	1.000	1.000
A	5	0.037	0.063	1.076
B	5	0.060	0.966	0.987
C	10	0.042	0.525	0.967
D	20	0.027	0.222	1.756
E	20	0.039	0.205	0.867
B	20	—	—	1.000

Il est important de remarquer que dans toutes ces expériences le métal n'est pas entièrement dissout. On trouve pourtant une récolte moins abondante quand la quantité de métal ajouté est plus grande. Nous avons observé ce résultat inattendu déjà avec le bismuth dans les mémoires précédents [1]).

4. *Différence entre l'action inhibitrice de germanium, d'étain et de plomb.*

On sait que le plomb ne possède aucune action inhibitrice sur la croissance du B. tuberculeux. Il est cependant utile de le comparer avec le germanium et l'étain. Le tableau 3 montre que l'action inhibitrice de germanium dans les groupes *A* et *B* est réellement plus élevée que celle d'étain. L'action inhibitrice de germanium est dans tous les cas examinés réellement plus élevée que celle du plomb.

L'action inhibitrice de l'étain est dans les groupes *A*, *D* et *E* réellement plus élevée que celle du plomb.

En résumé, l'action inhibitrice de germanium sur la croissance du B. tuberculeux est plus élevée que celle d'étain. L'action inhibitrice d'étain est plus élevée que celle de plomb. Ce dernier n'a practiquement aucune action inhibitrice.

5. *Repiquage du B. tuberculeux cultivé dans le milieu de culture contenant le germanium ou l'étain.*

En repiquant les cultures contenant le Ge ou le Sn après un temps déterminé sur un milieu de culture sans métal on peut constater deux

TABLEAU 3.

Différence entre l'action inhibitrice de germanium, d'étain et de plomb sur la croissance du B. tuberculeux.

Métal	Poids moyen en mg	t	P
	A. 5 mg de métal/100 cc		
Ge	25.93	2.346	0.06
Sn	43.75	94.301	$\ll 0.001$
Pb	750.50		
	B. 5 mg de métal/100 cc		
Ge	49.38	28.021	$\ll 0.001$
Sn	790.25	2.174	0.08
Pb	807.13		
	C. 10 mg de métal/100 cc		
Ge	33.40	1.838	0.1
Sn	420.08	1.634	0.2
Pb	773.75	55.392	$\ll 0.001$
Ge	33.40		
	D. 20 mg de métal/100 cc		
Ge	7.13	0.530	0.6
Sn	5.88	3.746	0.01
Pb	465.75		
	E. 20 mg de métal/100 cc		
Ge	24.20	0.877	0.4
Sn	127.33	3.305	0.01
Pb	539.75		

éventualités. Premièrement si le B. tuberculeux est encore vivant et en second lieu s'il montre une diminution de croissance dans le cas où il est encore vivant. Pour constater ce dernier cas il faut faire des cultures témoins sans métal.

L'expérience a été faite comme suit: on ensemence avec la souche Vallée$_{16}$ d'une culture de 13 jours, quatre ballons contenant 100 cc de milieu de Sauton sans métal (témoins), deux ballons contenant 100 cc de même liquide avec 40 mg de Ge et deux autres ballons contenant 100 cc de même liquide avec 40 mg de Sn. Après une culture de 7, 15 et 22 jours on fait de chaque culture témoin un repiquage et de chaque cuture à Ge ou à Sn deux repiquages sur milieu de Sauton sans métal. On obtient ainsi de chaque série quatre repiquages. Après un séjour de 25 ou 26 jours à l'étuve on détermine le poids sec de ces cultures secondes.

Le résultat (tableau 4) montre que, le B. tuberculeux cultivé dans le milieu de culture avec Ge ou Sn n'est pas tué. En comparant les poids moyens des cultures secondes avec celles des cultures témoins on peut

TABLEAU 4.

Repiquage du B. tuberculeux cultivé dans un milieu de culture contenant 40 mg de *Ge* ou de *Sn* par 100 cc.

Repiquage après *n* jours	Durée de l'expériene en jours	Cultures témoins Poids moyen en mg	Cultures à Ge Poids moyen en mg	Cultures à Sn Poids moyen en mg
7	25	573.63	129.25	105.70
15	26	738.58	593.75	231.00
22	25	685.83	424.20	199.25

savoir s'il existe une diminution de croissance (tableau 5). Il en résulte que les cultures secondes obtenues à partir des cultures à Ge ou à Sn

TABLEAU 5.

Comparaison des cultures de repiquage.

Cultures	Repiquage après 7 jours			Repiquage après 15 jours			Repiquage après 22 jours		
	$\bar{p}$	t	P	$\bar{p}$	t	P	$\bar{p}$	t	P
Témoins	573.63			738.58			685.83		
Ge	129.25	6,081	0.001	593.75	5.552	0.003	424.20	4.162	0.004
Sn	105.70	6,341	$<$ 0.001	231.00	18.280	$\ll$ 0.001	199.25	22.745	$\ll$ 0.001
Ge	129.25	0.672	0.5	593.75	10.645	$\ll$ 0.001	424.20	3.040	0.04
Sn	105.70			231.00			199.25		

donnent un poids moyen réellement inférieur à celui des cultures témoins. Il y a donc une diminution de croissance.

On constate de plus que le repiquage après 7 jours ne donne pas de différence réelle entre les cultures secondes de Ge et de Sn. Par contre, des repiquages effectuées après 15 et 22 jours montrent une différence réelle entre les cultures secondes de Ge et de Sn. Le poids moyen des cultures secondes de Sn est dans ce cas toujours inférieur à celui des cultures secondes de Ge. On pourrait l'expliquer que la lésion du B. tuberculeux provoquée par l'étain est plus profonde que celle provoquée par le germanium.

Nous avons vu que l'addition de germanium à un milieu de culture donne une action inhibitrice plus élevée que celle d'étain. Le germanium a donc qualitativement une différente action que l'étain.

Un phénomène intéressant a été observé dans les cultures secondes de Ge. Déjà deux jours après l'ensemencement on constate à la surface du liquide une voile mince. Après 17 jours et peut-être plus tôt la surface du liquide est totalement couverte par une voile mince de B. tuberculeux. De plus la voile monte contre la paroi du ballon jusqu'à 26 mm. En ce moment la surface de liquide des cultures témoins n'est pas encore totalement couverte. Les cultures secondes de Sn montrent également ce phénomène, mais il est moins prononcé.

6. *Addition de germanium ou d'étain métallique au cours de la croissance du B. tuberculeux.*

En ajoutant du germanium ou de l'étain à une culture de B. tuberculeux on peut voir si la croissance est arrêtée ou bien si la croissance est diminuée. Dans le premier cas on compare à la fin de l'expérience le poids moyen des cultures avec métal avec celui des cultures témoins du même âge $\bar{p}_i$ au moment de l'addition de métal. Dans le second cas on compare à la fin de l'expérience le poids moyen des cultures avec métal avec celui des cultures témoins du même âge, $\bar{p}_{36}$.

Pour chaque métal et pour les témoins on prend quatre cultures. On ensemence avec la souche Vallée$_{16}$ d'une culture de 25 jours. Durée d'expérience 36 jours. Le tableau 6 montre que l'addition de 40 mg de Ge

TABLEAU 6.

Addition de 40 mg de *Ge*/100 cc pendant la croissance du B. tuberculeux.

Temps en jours	Cultures témoins		Cultures à Ge	Comparaison avec $\bar{p}_{36}$		Comparaison avec $\bar{p}_i$	
	$\bar{p}$	τ	$\bar{p}$	t	P	t	P
15	96.50	3.7	140.88	13.399	$<<0.001$	2.071	0.09
22	356.95	7.3	705.75	6.238	<0.001	0.586	0.6
29	694.75	71.2	747.88	0.089	>0.9	3.153	0.02
36	743.78						

$\bar{p}$, poids moyen; τ, temps de génération.

par 100 cc après 15 et 22 jours ne donne pas de différence significante avec les cultures témoins au moment de l'addition de Ge. Les cultures sont donc practiquement arrêtées. L'addition de Ge après 29 jours n'a aucune influence, les cultures continuent à développer. Il n'y a pas de différence entre les cultures et les cultures témoins p_{36} du même âge. On pourrait supposer que pendant la croissance logarithmique le B. tuberculeux est plus sensible à l'action de Ge que pendant la croissance ralentie. En effet, on peut constater dans les mémoires précédents [1]) que la croissance logarithmique est pratiquement terminée au bout de 29 jours.

On constate dans le tableau 7, que l'addition de 40 mg de Sn par 100 cc ne donne aucun résultat. Les cultures continuent à développer,

TABLEAU 7.

Addition de 40 mg de *Sn*/100 cc. pendant la croissance du B. tuberculeux.

Temps en jours	Cultures témoins $\bar{p}$	Cultures à Sn $\bar{p}$	Comparaison avec $\bar{p}_{36}$		Comparaison avec $\bar{p}_i$	
			t	P	t	P
15	96.50	711.75	0.522	0.6	11.896	$<<0.001$
22	356.95	731.03	0.268	0.8	10.938	$<<0.001$
29	694.75	735.70	0.174	0.9	2.195	0 08
36	743.78					

la différence de poids avec les cultures témoins n'est pas significante. Il serait peut-être possible d'obtenir une différence de poids significante, si nous avions utilisé pour l'ensemencement une vieille culture. En effet, on constate dans les mèmoires précédents [1]) qu'une vieille culture est plus sensible à l'action de Bi qu'une jeune culture.

7. *Différence d'action de germanium et d'étain ajoutés au milieu de culture au cours de la croissance du B. tuberculeux.*

L'expérience précédente de Ge et de Sn étant effectuée en parallèle, on est donc en droit de comparer l'action de germanium à celui d'étain. Le calcul obtenu est reproduit dans le tableau 8. On constate que l'action inhibitrice de germanium est plus élevée que celle d'étain si l'on ajoute le métal au 15ème jour. Par contre, l'addition de germanium au 22ème et au 29ème jour ne donne plus de différence avec l'étain.

TABLEAU 8.

Différence entre l'action inhibitrice de germanium et d'étain ajoutés pendant la croissance de B. tuberculeux.

Temps en jours	Cultures à Ge Poids moyen en mg	Cultures à Sn Poids moyen en mg	t	P
15	140.88	711.75	13.220	$\ll 0.001$
22	705.75	731.03	0.501	0.6
29	747.88	735.70	0.658	0.5

On peut en conclure de l'expérience précédente que l'addition de germanium au cours de la croissance du B. tuberculeux peut arrêter la culture, si l'on ajoute le métal au moins de 22 jours. Par contre, l'addition d'étain ne donne aucune action dans tous les cas examinés.

L'action inhibitrice de Ge, de Sn et de Pb sur la croissance de B. tuberculeux diminue à mesure que le nombre atomique augmente, Ge (32), Sn (50) et Pb (82). Par contre, l'action inhibitrice de As, Sb et Bi augmente en même temps avec le nombre atomique, As (33), Sb (51) et Bi (83). [1]) Dans les deux cas l'action inhibitrice de ces éléments est en accord avec le classement des éléments dans le tableau périodique.

8. *Comparaison de l'action inhibitrice de germanium, d'étain et de plomb avec celle d'arsénic, d'antimoine et de bismuth.*

Les données expérimentales obtenues dans les mémoires précédents et celles dans ce mémoire permettent de faire une comparaison quantitative entre l'action inhibitrice de Ge et de As, de Sn et de Sb et enfin de Pb et de Bi. Pour cela les poids de microbes obtenus avec 10 mg et 20 mg de métal sont rapportés au poids moyen des témoins. Ce dernier étant égal à 1. On peut ensuite savoir si la différence entre les poids moyens obtenus avec les métaux est significante. Le tableau 9 montre le résultat des calculs. On constate le même résultat avec 10 mg et 20 mg de métal. La différence d'action entre Ge et As d'une part et entre Pb et Bi d'autre

part est significante. L'action de Ge (32) est réellement supérieure à celle de As (33), par contre celle de Pb (82) est réellement inférieure à celle de Bi (83). Il n'existe pas de différence réelle entre l'action inhibitrice de Sn (50) et de Sb (51). Ce résultat est en accord avec la conclusion donnée plus haut.

TABLEAU 9.

Comparaison de l'action inhibitrice de germanium, d'étain et de plomb avec celle d'arsénic, d'antimoine et de bismuth.

Métal	Poids moyen	*t*	*P*
10 mg de métal/100 cc			
Ge	0.042	16.698	$<<0.001$
As	0.911		
Sn	0.525	1.296	0.3
Sb	0.578		
Pb	0.967	19.066	$<<0.001$
Bi	0.024		
20 mg de métal/100 cc			
Ge	0.033	3.107	0.01
As	0.259		
Sn	0.113	0.346	0.7
Sb	0.066		
Pb	1.208	5.332	<0.001
Bi	0.066		

Résumé.

1. Le germanium et l'étain ont une action inhibitrice sur la croissance du B. tuberculeux.

2. L'action inhibitrice de germanium est supérieure à celle d'étain. Le plomb n'a practiquement aucune action inhibitrice. Ce résultat concorde avec le classement de ces éléments dans le tableau périodique.

3. Le B. tuberculeux n'est pas tué après une action prolongée de germanium ou d'étain. Le repiquage donne cependant une culture moins développée, l'action d'étain est supérieure à celle de germanium.

4. Si l'on ajoute le germanium pendant la croissance du B. tuberculeux, la croissance est arrêtée. Dans ce cas l'étain n'a aucune action.

5. L'action de germanium sur le B. tuberculeux diffère qualitativement de celle d'étain.

6. L'action inhibitrice de germanium, d'étain et de plomb diminue à mesure que le nombre atomique augmente: Ge (32), Sn (50) et Pb (82). Par contre, celle d'arsénic, d'antimoine et de bismuth augmente en même temps avec le nombre atomique: As (33), Sb (51) et Bi (83).

7. L'action inhibitrice de Ge (32) est supérieure à celle de As (33). Par contre, celle de Pb (82) est inférieure à celle de Bi (83). Il n'y a pas de différence entre l'action de Sn (50) et celle de Sb (51).

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Botany. — *Contribution to a theory on the absorption of salts by the plant and their transport in parenchymatous tissue.* By W. H. ARISZ.

(Communicated at the meeting of November 24, 1945.)

§ 1. *Introduction.*

The problem to be discussed here, has to do with the ability of plant cells to take up salts from their environment. From vegetationexperiments of many investigators, such as TRUE and BARTLETT (1915), PARKER (1927), V. WRANGELL (1928), JOHNSTON and HOAGLAND (1929) and others, it has appeared that plants can absorb anorganic salts even from very dilute solutions. This renders it comprehensible that they can also take up an adequate quantity of food from the soil-solution and from ditchwater. Whereas it was originally thought that the salts are carried along by water that the plant takes up as a result of transpiration, it has become clear in later years that the uptake of salt is a complicated process, which though it may be more or less affected by water-absorption, for the rest takes place independently of it. So it comes to pass that also submerged waterplants, which naturally show no transpiration, yet take up nutrient substances as salts and aminoacids from the environment. This is partly done by their roots, partly by their leaves, as for instance Elodea and Vallisneria (ROSENFELS (1935), ARISZ).

Besides roots and leaves a third group of organs is to be mentioned, which under special circumstances are likewise able to take up salts. Among these are tubers and other organs containing reserve food, which have been cut into discs and as a result of the wounding at their surfaces possess actively synthetizing cells (STEWARD). Moreover there is a fourth group: organs which have the special function of taking up substances from their environment. Of these some instances may be mentioned: 1. The tentacles of the insectivorous Droseras, which have been extensively examined by OUDMAN and ARISZ and 2. the scales on the leaves of Bromeliaceae, the function of which has been investigated by A. HARBRECHT (1942). Finally the lower plants may be mentioned here, of which especially the unicellular algae have been repeatedly examined. (OSTERHOUT, BROOKS, HOAGLAND, COLLANDER and many others.) The processes in such unicellular organisms seem simpler, but they are less suitable for an analysis of the phenomena, because the processes of metabolism are much more intensive here and have a complicating influence on the uptake and transport-processes we are interested in. This makes the analysis by these unicellular plants much more difficult.

It is interesting to trace whether in the above cases we have to deal done by their roots, party by their leaves, as for instance Elodea and

with identical absorption processes. They have in common their dependence on the aerobic respiration of the plant. There is a difference, because the place to which the substance taken up is being carried and the route which is followed, are not the same for every case. In the leafcells of Vallisneria and Elodea and likewise in the discs of reserve organs, the salts chiefly go to the cellsap of the absorbing cells. In the root, however, only part of the substances taken up, will be fixed in the vacuoles of the absorbing cortical cells. A considerable part is carried to the wood-vessels in the central cylinder through the cortical parenchyma cells and the endodermis. Especially this latter process has to be considered as the proper task of the root. The well-known experiments of HOAGLAND and BROYER on the absorption of salts by the root have been made in such a way that the first process, in which the substances are secerned into the vacuoles of the cortical cells is much more striking than the second process, the transport to the xylem vessels. This is due to the way in which these experiments have been made. Excised root systems are used, in which the wood vessels have been opened, so that part of the substances given to them, is returned to the liquid nutrient. By a suitable preliminary treatment in a nutrient solution poor in salt, the tissue is low in salt at the beginning of the experiments, while care has been taken that there is a sufficient quantity of reserve food. The result is a strong accumulation in the cortical cells. Owing to this the results of their experiments show a strong resemblance to STEWARD's on accumulation by potato-discs. They are, however, not comparable with the experiments on the absorption of salt by various other investigators who worked with roots with normal salt content that had not been cut off, as for instance those by LUNDEGARDH, where the substances are chiefly given to the woodvessels and the accumulation in the cortical cells is a matter of secondary importance.

We shall revert to the analysis of these processes in the root, when discussing the permeability of the tissue and in § 7. Here we may point out that in all these cases absorption and loss of substances by the protoplasm go together. For though we usually speak of absorption of substances by the vacuole, it is more correct to consider this process from the angle of the protoplasm and to speak of secretion into the vacuole, especially if we have to deal with active processes here. On the one side the protoplasm absorbs the substance from the cell wall and the medium, on the other side releases it to the vacuole. This points to the fact that absorption and secretion are processes experimentally difficult to separate. The secretion of substances by the protoplasm externally, as it occurs in gland cells or in excretory organs such as salt glands and nectaries, may be considered a related process.

Not only are absorption and secretion of substances by the protoplasm processes which are closely connected, but to these may be added the transport of substance in the protoplasm, which can no more be strictly

separated from them. Every absorption and secretion is attended by a movement of the substance in the protoplasm, even in a unicellular organism, where the substance from the environment is carried through the cell wall and the surface zone of the protoplasm to the more inward part of the plasm, whence it can permeate through the tonoplast into the vacuole. With multicellular organisms the movement is more striking, for there the surface layer of the epidermal cells absorbs the substances, whence they are transported to the adjacent cells. Thus transport from cell to cell in the cell walls and the symplasm is possible in the parenchymatous tissue. In the roots too the substances have to be transported over a rather long distance in a radial direction. In the tentacles of Drosera the transport from the glands to the leaf by the cells of the tentacles comes to the fore. These are specialised transport organs (ARISZ 1944). Absorption, secretion and transport therefore are three processes, which are closely related and cannot easily be circumscribed.

§ 2. *Permeability, intrability and transmeability.*

From a great number of researches it has appeared that the absorption of salts into the vacuole occurs on lines entirely different from those followed by most organic substances. Only the dissociated organic substances, as the amino-acids, show some correspondence with the salts (ARISZ and OUDMAN 1938). Before proceeding to set forth the different behaviour of these substances, we have to discuss the terminology of the penetration of substances into the cell. To indicate that a substance passes through the plasm, we usually speak of permeation. The plasm is then called permeable to the substance. If like PFEFFER we consider a plant cell as an osmotic system, the semipermeable membrane is the boundary surface of the protoplasm. There are two of these boundary surfaces to the plasm, one bounding it on the outside and one contiguous to the vacuole. Through H. DE VRIES' fundamental researches, it has become possible to determine the permeability of the plasm for various substances as glycerine and ureum. The *permeability* of the protoplasm which was determined in these researches, therefore, comprised both boundary surfaces and the plasm between them. Later HÖFLER introduced the terms *intrability* and *intrable* to indicate that a substance permeates into the plasm through the surface zone, but cannot go through the tonoplast to the vacuole. Though this terminology is historically comprehensible, it may lead to misunderstanding and trouble. The difficulties are chiefly of two kinds. In the first place we speak of *permeability* by animal cells in which a vacuole is lacking and the same holds good of plant cells without vacuoles, when a substance penetrates the protoplasm. On comparing the permeability of vegetable and animal tissue or cells with and without vacuole, we consequently often compare heterogeneous quantities. The *intrability* of one kind would have to be compared with the *permeability* of another. In the second place confusion may arise

when a substance permeates through a tissue. This may happen without the substance permeating into the vacuoles of the cells. In that case the substance must permeate through the plasmalemma and move in the plasm without proceeding through the tonoplast, after which the substance can leave the plasm of the cell in order to continue its journey in an adjacent cell. If the representation is correct that the plasm of adjacent cells is connected by plasmodesmata, we have to do here with a transport in the symplasm. A fine example of this kind of plasm-permeability is found in the endodermis sheath in the root. According to DE RUFZ DE LAVISON various salts in a high concentration, which causes plasmolysis, permeate into the central cylinder. In doing so they have to pass through the plasm of the endodermis cells, as along the radial and cross walls no substances can be transported, because the cork-bands of CASPARY prevent this, whereas with plasmolysis of the endodermis cells the connection between plasm and cork bands is preserved. (cf. fig. 1.) In such a case we are entitled to speak of plasm-

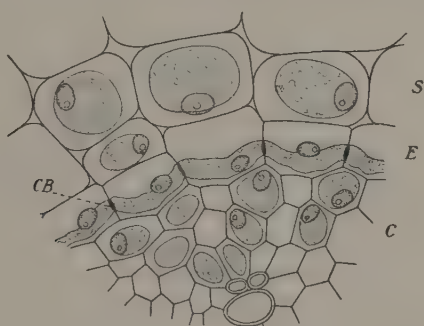

Fig. 1.
Transverse section of a plasmolysed root.
S = cortex. E = endodermis. C = stele. CB = Band of CASPARY.
The plasmolysing solution in penetrating the stele has to pass the plasm of the cells of the endodermis.
(From RUFZ DE LAVISON.)

permeability. This is no intrability, because the substance leaves the plasm again. We may arrive, however, at entirely wrong conclusions, if we compare the results obtained in this way for permeation through the plasm, with those obtained for permeation of the substance through the plasm from environment to vacuole, in which case the tonoplast has to be passed too. In order to avoid confusion, it is desirable to indicate one of the two processes by an other name. In order to maintain the word permeability for cells possessing no vacuole, such as all animal cells, it is desirable to give a different name to the penetration from the environment through the protoplasm into the vacuole. In this case we shall use the terms *transmeability* and *transmeation*. *The protoplasm, therefore,*

that allows a substance to pass through both membranes, through plasmalemma and through tonoplast will be called transmeable here. We then arrive at the following definitions. The general conception penetrable will be expressed by *permeable*. This may obtain for the wall, the plasm or for plasmalemma and tonoplast separately. If an exact indication of the kind of permeation through the plasm is desired, we use *intrable to designate that the substance permeates through the plasmalemma into the plasm and transmeable when the substance passes through plasmalemma as well as tonoplast to arrive in the vacuole.* If the substance does not pass through the tonoplast, but leaves the plasm through the plasmalemma, we speak of *permeable sensu stricto*.

If we abide by this terminology, it may be prevented better than has been the case hitherto, that processes of various nature are put on a level. If we consult summarizing works on permeability, as e.g. Das Permeabilitäts Problem by GELLHORN, it strikes us that not even an attempt has been made to separate the data on intrability from those on transmeability. Indeed the investigators themselves repeatedly leave it undecided which quantity they have determined. This for instance holds good of those, who compare data on Bacteria and other unicellular organisms, which possess no or only small vacuoles with those on higher plants where transmeability has been determined. It may be imagined that in some cases it may occur that the tonoplast is better permeable than the plasmalemma. In that case it will make no difference whether an investigation is made into intrability or transmeability. Instances of cases in which this has actually been investigated, are certainly not abundant. The same objection obtains for a survey by KAHO (1926) in ‚die Ergebnisse der Biologie', where data on intrability in higher plants are being compared to those on transmeability without it being considered that this is not permitted without comment. Also LUNDEGARDH's method (1911) to determine the permeability of the root from the change in length of its apex by permeation of substances, deserves further study, because we are not sure which quantity was determined in these young cells rich in plasm. With this type of experiments it is possible that in older cells transmeability is determined, in the younger cells intrability. The meaning of the terminology proposed here, is to get a better base for comparative observations.

We shall now proceed with the discussion of the permeation of various substances. We already stated that dissociated substances behave differently from non-dissociated ones. About the permeation (here: transmeation) of this latter group of substances we have obtained good information through the researches of HÖFLER and COLLANDER with their collaborators. We know that they can more or less penetrate into the vacuoles of the living cells through diffusion and that this process is dependent on the ‚transmeability' of the protoplasm. The permeation into the vacuole and the exosmosis of the absorbed substance from the vacuole

when the cells are again submerged in water, are diffusion-processes which proceed in the same way in either direction. Transmeation through the protoplasm is determined by the size of the molecules and by the lipoid-solubility of the diffusing substances. In case of equilibrium the substances in the vacuole will be present in the same concentration as in the medium, unless the solubility of the substances in the cell sap is different from that in the external solution or the substances are being chemically converted or fixed. This may cause an accumulation of substance in the vacuole, but it is of a nature quite different from the accumulation which often occurs on accumulation of salts, when the substances are piled up in the vacuole unaltered and are found there in a dissolved condition.

For salts too it is assumed by various investigators that they permeate through the protoplasm. Especially in Beggiatoa mirabilis a great permeability to salts was found by RUHLAND. The researches by FITTING, WEIXL HOFFMAN, JARVENKYLA and MARKLUND have shown that in higher plants salts can permeate through the plasm in a slight degree (permeate used here in the sense of transmeate). It remains, however, uncertain whether in experiments with non-balanced solutions we have to deal with a normal behaviour of the protoplasm. Besides we have to consider that these experiments have often been made with concentrated or fairly concentrated salt solutions, so that they do not prove that very low concentrations, such as act a part in the environment, can pass as well. Objections to plasmolysis-experiments have been repeatedly been raised (RUHLAND, ILJIN, STILES, SCHMIDT, ARISZ and VAN DIJK). We shall see in the next section that there is reason to surmise that from such low concentrations there can penetrate ions into the protoplasm, but that a passive diffusion of dissociated or non-dissociated salt molecules in the vacuole is improbable. This makes us suppose that the positive results obtained with high concentrations of a one-salt-solution might be connected with modifications in the conditions of the protoplasm, which are brought about by the abnormal concentrations of certain ions, after the ions have penetrated into the protoplasm through the intrable outer layer. It is known that one-salt-solutions, such as are used for plasmolysis-experiments, are extremely poisonous as nutrient solutions. Moreover by the use of high osmotic concentrations the plasmolysis method causes alterations in the protoplasm which hinder its normal function, which according to ARISZ and VAN DIJK appears from the fact that the active uptake of asparagine is considerably hampered. In still unpublished researches the great sensitivenous of the active salt uptake has appeared for one-salt-solutions, even in a very dilute solution.

In correspondence with this are the experiences with „Kappen"-plasmolysis, a phenomenon by which the cytoplasm swells strongly. According to HÖFFLER this swelling is brought about by an enhanced intrability of the protoplasm due to the concentrated salt-solution. Different researchers

among whom SEIFRIZ, LEPESCHKIN and HÖFLER assume that the plasmalemma allows ions to pass, the tonoplast does not. Solutions so highly concentrated as are used for permeability-experiments, however, don't act a part in the environment of most organisms. By the absorption of salts from ditch-water or from the soil we have to deal with much lower concentrations 1/10000 to 1/100000 mol and on the question whether such low concentrations permeate or not plasmolysis-experiments can throw no light. To be sure, in the organisms living in the sea the salt-concentrations in the environment are much higher, but it is surprising to find that the composition of the cell sap also in unicellular weeds living in the sea may considerably differ from that of the environment (OSTERHOUT), which surely does not suggest permeability of the plasm for these ions. On the ground of his experiments with unicellular weeds COLLANDER has arrived at the conviction that the plasm as a whole does hardly allow any salts to pass and one should rather be surprised that the protoplasm is so little permeable to salts. By the very impermeability of the protoplasm maintaining a high osmotic concentration in the vacuole is possible. So when we notice that substances which do not permeate are yet accumulated in the vacuole this points to the existence of a process through which these substances are taken up from the environment into the protoplasm and are passed on by the protoplasm to the vacuole. Our intention is to analyse this process of absorption in this article. It may be divided into the following parts which will be treated separately.

They are: 1. the uptake of ions from the environment into the protoplasm.
2. the secretion of ions by the protoplasm into the vacuole.
3. the transport of ions in the protoplasm and from cell to cell.

It is not our intention to give an extensive discussion on the mechanisms of the interchange of ions, of the transport in the protoplasm and of the accumulation in the vacuole. It may, however, be valuable to analyse the combined action of the various processes and to throw some light on some essential points in the salt-absorption of the root.

§ 3. *The uptake of ions into the protoplasm. The permeability of the surface layers of the protoplasm.*

From the great number of experiments made in the last few years with radio-active ions, it has become clear that they are easily taken up into the plant. If a plant with roots is placed in a medium with radio-active ions, their presence in the root can be demonstrated after a short time. Next they penetrate into the wood vessels and they are rapidly spread through the whole plant with the transpirationstream. STOUT and HOAGLAND also found a strong lateral transport through parenchyma-tissue from the wood to the cortex.

From experiments by JENNY, OVERSTREET and AYERS with radio-active ions it appears that the outer layer of the protoplasm is penetrable for ions. If a plant which has taken up radio-active potasium ions, is trans-

ferred to a solution without radio-active potassium ions, there ensues a loss of radio-active potassium ions, if a potassium salt is present in the solution, which is, however, not the case when the medium is pure water or a solution of salts with other cations, as sodium, This points to an interchange of radio-active potassium ions and non-radio-active ones, and it may be concluded from this fact that under normal circumstances potassium ions from the medium are continuously interchanged with potassium ions from the plant. The surface zone of the protoplasm is therefore permeable to these ions. The same obtains according to these investigators for radio-active sodium- and bromine ions. If therefore a plant does interchange radio-active potassium against other potassium-ions, but not against sodium-ions, which are present in the medium, this is not due to the fact that the plant does not allow these ions to pass. but because the conditions for interchange are not present. From this it may be concluded that *the surface layer on the outside of the protoplasm renders interchange of ions possible*. This may be called *interchange-permeability of the surface-layer.*

OVERSTREET and BROYER investigated into the uptake of radio-active potassium in barley. This may be an active absorption in the sense of STEWARD and HOAGLAND or a cation interchange. At 0° C. radio-active potassium is taken up as well, but as at this temperature no active absorption can take place, the uptake must be entirely based on the interchange of cations under these circumstances. This interchange proved to continue till a limit is reached at which 10 % of the total quantity of potassium present in the roots has been interchanged. The remainder of the potassium is not interchanged. The writers state that the interchangeable potassium "is believed to be associated with the colloidal phases of the protoplasm and cell wall". From this it follows in my opinion that the potassium present in the vacuoles cannot be interchanged at 0° C. If one conceives that the protoplasm of the parenchymatous cells forms one coherent whole, which deserves the name of symplast, this means that from the whole symplast potassium ions can interchange with ions of the medium, but that the potassium present in the vacuoles is not interchangeable.

This makes us surmise that in these experiments the *vacuole-bounds are impermeable to potassium-ions*. It was, however, shown by JENNY and OVERSTREET that in a medium of acid clay colloids (pH 2.9—3.5) the potassium from the vacuole can be replaced by hydrogen as well, because they found that as much as 20 % and even more of the potassium present can be interchanged with H-ions. From this they concluded that all phase-boundaries are permeable to cations in both directions. They, however, inform us that in experiments with acid clays the cells are damaged, so that the permeability of the tonoplast may be founded on its damaged condition. In less acid clays the cells were not damaged irreversibly. Yet in our opinion normal physiologic conditions are out of the question in

these experiments as well, so that it seems possible that here too under the influence of the high concentration of the H-ions in the medium the conditions of the protoplasm and of the surface-layer of the vacuole have been changed in such a way that the tonoplast is no more capable of checking ions.

In connection with these researches we shall discuss here a research of MAZIA's on Elodea, from which it also appears that the tonoplast as contrasted with the plasmalemma is impermeable to certain ions. In Elodea no calcium is present in the vacuole, while calcium that gets into the vacuole immediately crystallizes as calciumoxalate. Indeed there is calcium in the protoplasm; this may be removed from the cell by potassium citrate, without its being irreversibly damaged. If it is subsequently brought into a solution of calciumchloride, the plasm again absorbs calcium ions. The presence of calcium in the protoplasm may be demonstrated by the fact that with certain stimulation reactions (MAZIA and CLARK) calcium is released to the vacuole and crystals of calcium oxalate are formed. The vacuole wall therefore is in this case under normal circumstances impermeable to calcium ions, while the plasm can easily deliver and absorb calcium ions through interchange with other cations Mg, Sr, Ba, K, Na in the medium (MAZIA 1938). These data point to the fact that interchange of ions between *plasm* and *medium* can easiliy take place, but that the *vacuole* cannot allways participate in it, because the tonoplast does not allow all ions to pass. MAZIA, however, like HEILBRON assumes that as a result of the stimulation the plasm sets free Ca-ions, so that they diffuse in the vacuole.

BROOKS and BROOKS (1941) also hold that ions easily penetrate into the protoplasm, provided there are other ions that leave the plasm at the same time. Both ions in the surface zone and ions from the whole plasm, which are combined with proteins there, participate in this.

BROOKS found a similar interchange of ions by Nitella for potassium and rubidium, but not for bromine. He stated (1937) that if a Nitella cell is brought into a solution of radio-active potassium, for the first six hours no radio-active potassium penetrates into the vacuole, whereas already after a few minutes radio-active potassium has penetrated into the protoplasm through interchange of ions. It seems to me that from this fact it may also be concluded that between *vacuole* and *protoplasm* interchange of ions does not occur or with great difficulty. BROOKS himself, however, assumes that ions get from the plasm into the vacuole by a concentration gradient through diffusion. All the above researches therefore support in our opinion the *conception that the protoplasm and its surface zone are permeable to ions, but that certain ions cannot pass into the tonoplast.* Yet we see in some cases that exosmosis of actively absorbed substances from the vacuole occurs. If for instance the protoplasm and the tonoplast are in abnormal circumstances, the surface zone of the vacuole may temporarily become permeable to ions, so that exos-

mosis of ion pairs or interchange of cations and anions may take place. Such an exosmisis was found (ARISZ 1943) as a reversible phenomenon in Vallisneria leaves when after taking up asparagine, they are transported to another solution, pure water or a fresh asparagine solution, in which there are not any salts. Evidently in order to remain in an active condition the protoplasm with its two surface layers needs a medium of a definite composition in which various cations and anions have to be present in an extremely low concentration. By release of ions to the medium the normal condition of the protoplasm is restored after some time; then the protoplasm is again capable of taking up asparagine. This conception corresponds with the data in literature on the toxicity of distilled water and of salt solutions and agrees with the views on antagonism of ions of SEIFRIZ, LEPESCHKIN and HÖFLER.

It has appeared from different experiments that besides through interchange ions can also be taken up without ions of the same charge being released at the same time (LUNDEGARDH, HOAGLAND). In this case in order to maintain the electric equilibrium, ions with an opposite charge must be taken up, either as ion pairs or as BROOKS and LUNDEGARDH suggest, because at certain points cations and at other points anions enter. Now the question will have to be answered whether ions can be taken up into the plasm to an unlimited amount. It is probable that they are partly free, partly bound to the plasm. This binding will probably be of a chemical nature, but it behaves as an adsorption binding. For convenience sake we talk in this case of *adsorbed* ions. All ions in the plasm, except of course those which are built-in in stable chemical compounds, that is both adsorbed and free ions, can be interchanged with ions from the medium.

The concentration of the free ions present in the plasm, is probably determined by Donnan equilibria. In addition the distribution of ions between plasm and medium will also be influenced by the "diffusion effect", studied by TEORELL (1935), in consequence of the diffusion outward of a dissociated substance. The ions in the protoplasm cannot be present in a higher concentration in a free state than the one fitting these equilibria. Otherwise they would leave the cell through the outer layer, which is permeable to ions, whereas from numerous researches it appears that for instance to distilled water no or exceedingly few ions are released.

In order to continue the uptake of ions from the medium as a continuous process, the ions adsorbed to plasmic particles in the surface zone should be removed and leave room for the binding of other similar ions from the medium. From the experiments of JENNY and others with radioactive ions it appeared that simultaneously ions from the plasm go to the medium and vice versa. Owing to the removing of ions from the plasm at a constant speed a streaming of ions will result from the medium to the protoplasm.

This conception of the uptake of ions corresponds very well with the quantitative data on the strength of ion absorption from solutions of various concentrations. In the most varying objects it has always been found that from low concentrations the uptake is relatively greater than from high ones. The curve showing the connection between the concentration of the absorption and the concentration of the solution in the medium has the course of the adsorption isotherm of FREUNDLICH. Various investigators have considered this as an indication that the substances in the cell would be bound by adsorption (STILES, LUNDEGARDH). Others (HOAGLAND, COLLANDER) on the contrary believed that the substances are present in the vacuole in a free state. SCARTH and especially VAN DEN HONERT pointed out that this course of the curve can likewise be explained by the fact that an adsorption process has a limiting effect on the uptake.

So in his experiments on the uptake of phosphates VAN DEN HONERT comes to the conclusion that the ions are adsorbed at the surfacezone and are removed to the vacuole at a constant speed. The amount of ions taken up is on the one side determined by the amount of ions adsorbed at the surface zone, on the other side by the speed of removal. For this purpose VAN DEN HONERT makes use of the image of the conveyor-belt which had already been used by VAN DER WEY for the transport of growth substance before.

The process of the transport of ions in the protoplasm will be discussed in § 5. Below only the first part of the process is treated, the adsorption of the ions to the protoplasm. From the above it may be concluded that there is a tendency to attain an adsorption equilibrium between the ions which are bound in the outward layer of the protoplasm and the active concentration of the ions in the medium. Such an adsorption-equilibrium will generally be rapidly achieved, so that this process develops in most cases so quickly that the adsorption-equilibrium is approached. LUNDEGARDH (1941) found that the formation of the electric potential at the root surface for H-ions was reached in 0.75 sec., while on exchange of a 1/10000 into a 1/1000 m. salt solution, the formation takes place in 1 to 4 seconds, dependent upon the nature of the cation. This longer duration would indicate a diffusion over a distance of 2.9 μ as far as the adsorbing surface zone.

Various investigators have found the laws of adsorption applicable to the uptake of ions by the plant. We only mention STILES 1924, SCARTH 1925, LEMANCZYK 1926, NIKLEWSKI, KRAUSE and LEMANCZYK 1928, WRANGELL 1928, LUNDEGARDH 1932, 1935, 1938, 1940, PERIS 1936, LAVOLLAY 1936. Our own researches on the uptake of substances by the tentacles of Drosera and those on the uptake of substances by Vallisneria pointed to the significance of adsorption processes as well. This can therefore be based on the view that *the first phase of the process of uptake is an adsorption to the plasm and that the concentration of adsorbed ions*

is one of the factors determining the strength of the transport of ions

So in the preceding discussion we arrived at the conclusion that the outer layer of the protoplasm allows ions to pass, whereas the tonoplast may inhibit the passage of certain ions. The question may now be raised whether only ions which are adsorbed by the plasm can penetrate into the protoplasm, or that also ions can be taken up without being adsorbed to the surface zone. No direct data bearing on this subject are known. Though the pores will not be very narrow, because, as will be subsequently discussed, the protoplasm of leaf- and root cells is intrable for sucrose (see pag. 21) it is conceivable that owing to their charge the free ions cannot penetrate through the pores, whereas they will be able to do so, when they are adsorbed by the plasm (cf. PFEFFER and SCHÖNFELDER), because in that case they have a larger part of the pore at their disposal. *On the ground of the above representation that an adsorption process has a limiting influence on the uptake, it is likely that only adsorbed ions can be taken up.* If free ions also penetrated this relation would be impossible. Whether the binding of the inward-directed ions to the plasm already occurs in the surface zone or in a layer lying more inward, cannot be decided and does not matter here. Since in the protoplasm both cations and anions are bound, they may be simultaneously moved in the protoplasm and secreted into the vacuole. The behaviour of the ions discussed here only obtains for very low concentrations, as required for experiments with nutrient solutions. In higher concentrations and especially in such high concentrations as are used in plasmolysis experiments, the physico-chemical properties of the protoplasm will be modified by the entering ions. This gives rise to phenomena as "Kappen"-plasmolysis. Owing to the higher concentration of the medium a new equilibrium of the ions in the protoplasm will be formed. If a one-salt-solution is used, the relative ratio of the different ions in the protoplasm will be altered. As for a normal functioning of the protoplasm the ratio of the different ions must remain within certain limits, the physico-chemical condition will change, when these limits are exceeded. In that case the permeability of the tonoplast may alter likewise. Hence that in plasmolysis experiments results may be obtained about the permeability of the protoplasmic membranes which are different from those obtained in experiments with lower salt concentrations.

If the above conception of the permeability of the protoplasm and its membranes is correct, the accumulation of substances in the vacuole is in many cases an active secretion. The energy required for this process is provided by aerobic respiration. Of course this does not alter the fact that also in the protoplasm in the way indicated by TEORELL and also through chemical reactions substances may be accumulated.

§ 4. *The secretion of ions into the vacuole. The accumulation process.*

In many experiments it is impossible to determine with certainty wether

a substance that has been taken up by a cell, is present in the plasm, in the vacuole or in both. In fact this can only be ascertained for those cells that possess so large a vacuole that the cell-sap can be analytically examined. Only in a few cases the presence of a substance may be concluded from a chemical reaction, which brings about a visible alteration. Therefore experiments with weeds consisting of large cells such as the Characeae, Chara and Nitella and also with Valonia and Halocystis are of great value, because in those the cell sap can be analytically examined. STEWARD, however, drew attention to the fact, and also COLLANDER's observations are in accordance with it, that these cells show a relatively slight accumulation. Though BROOKS contests STEWARD and MARTIN's conception that Valonia is little active, by pointing out that their metabolism is active in proportion to the amount of protoplasm, this does not alter the fact that active accumulation in these cells is slight with respect to the volume of the sap. The experiments with these organisms, however, indicate that by active processes substances can be accumulated in the vacuole. For cells of a tissue we must do with less accurate methods. The investigators who analyse expressed sap from tissues in order to trace whether substances have been taken up, have to face the difficulty of proving that this expressed sap, at least in the main, corresponds with the vacuole sap. It appeared that especially in the last few years there was no unanimity on this head. It may, however, be expected that in cells with little protoplasm the sap, which is being expressed, after the protoplasm has been killed, will in the main correspond with the vacuole sap (STEWARD 1932, BROYER and HOAGLAND 1940).

A third method to trace whether substances are taken up in the vacuole, is the simultaneous determination of the osmotic value of the cell sap and of the amount of substance taken up in the cell. In tracing this a good correspondence between increase of osmotic value expected and found was ascertained in some cases (ARISZ and VAN DIJK, ARISZ 1943). Neither does this method give any certainty, because the change of osmotic value may as well be caused by ana- or cata-tonosis.

On the ground of researches with radio-active ions BROOKS arrived at the conclusion that Nitella first accumulates ions in the protoplasm and does not pass them to the vacuole until later. The accumulation in the vacuole therefore would be a result of a previous accumulation in the protoplasm and would not take place contrary to the concentration-gradient. COLLANDER could not corroborate these data by Nitella. A priori it does not seem probable that in the protoplasm an unlimited accumulation of freely diffusing ions could be maintained, while the outer layer of the protoplasm allows ions to pass. The view we developed in the preceding section that the tonoplast does not allow ions to pass, is conditional for the maintenance of salts in a higher concentration in the vacuole than in the protoplasm and the medium.

Also if one assumes, as has been done here, that for accumulation of

substances in the vacuole a concentration-gradient plasm-vacuole, is not required, the accumulation mechanism will have to be localized in the protoplasm contiguous to the vacuole, and one will after all be able to agree to the conception of BROOKS: "The concept which we wish to bring out, is that the protoplasm is the agent which is important in accumulating electrolytes".

On the nature of the accumulation mechanism, i.e. the uptake of ions by the vacuole various investigators have given theories, see among others BRIGGS, OSTERHOUT, LUNDEGARDH and HOAGLAND and BROYER's criticism (1940). We shall not go further into this in this article and consider this process either a consequence of the accumulation in the protoplasm or a secretion into the vacuole, in the way the protoplasm secretes substances to the vessels or to the medium (active hydathodes). Energy is needed for this performance.

The consequence of this accumulation must be further discussed here. For the organism it means that the osmotic value of the vacuole is being enhanced. For the growth of the cells this is essential, because the pumping in of osmotic substances into the vacuole of growing cells maintains a turgor pressure which is conditional for cell-elongation. This moreover depends on the presence of a number of growth factors, as growth substance and cell building material (cf. pag. 23). To the essential significance of this secretion process BURSTRÖM and FREY-WYSSLING recently also drew the attention.

When the question is asked whether the accumulation in the vacuole is of any consequence for the transport of substances in the tissue of the plant, this question must be answered in the negative, for it seems to be a not very efficient mechanism for this purpose. For instead of making the substances available for transport, they are fixed in the vacuole. The data on root systems of plants of high or low salt-concentration (HOAGLAND and BROYER) show, however, that the accumulation in the vacuole does not continue in an unlimited way and in connection with the enhanced osmotic value there may exist a limit, at which the accumulation decreases or stops. This puts an end to the accumulation in the vacuole, so that the substances become available for the adjacent cells in larger quantities. In this connection it is interesting to point to a supposition we made regarding the transport in the Drosera tentacles (ARISZ 1944). In these typical transport organs the vacuoles would be nearly put out of use through the aggregation of the cells of the tentacles, while the plasm through swelling takes up a volume as large as possible. With these specific transport cells therefore no or hardly any accumulation of the transport-substances into the vacuoles of these cells would take place. For transport purposes the cytoplasm is of pre-eminent importance.

§ 5. *The transport of ions in the protoplasm.*

The experiments with radio-active ions prove that the ions not only

penetrate into the cells contiguous to the medium, in which the radio-active ions are present, but that they are also easily transported from cell to cell and in consequence of this have been spread over large pieces of parenchymatous tissue after a short time. A transport of ions from cell to cell is therefore possible. Now that we have seen that the surface zone of the protoplasm allows ions to pass for interchange and the inner-layer, the tonoplast, may be impermeable to ions, only cell wall and cytoplasm are to be regarded for this transport of ions.

The experiments with radio-active ions are not the only examples of such a transport. We know that also under the influence of clays saturated with bases, in which the active concentration of the cations at the surface is very high (JENNY), interchanges of ions with large tissue-complexes take place. With these experiments it cannot be doubted that the ions which are present in the plasm are replaced by ions from the medium. In the experiments of various investigators, regarding the excretion by the roots of ions coming from the shoot (PRIANISCHNIKOW, ACHROMEIKO, SCHMIDT, LUTKUSS and BÖTTICHER) this process must take place through large strands of parenchymatous tissue. It is not known how far the release of ions from the above mentioned parts is based on a transport along special tracks (phloem), but partly this will no doubt be a transport through parenchymatous tissue. Also the radial transport of radio-active ions, which STOUT and HOAGLAND found in the stalk, must be transport in parenchymatous cells.

These phenomena therefore make the impression that through the plasm ions can be easily moved along fairly long distances and it seems likely that the adjacent cells are bound by plasmatic connections and form a symplasm (MÜNCH). *In the symplasm ion transport can easily take place.*

If, however, we don't hold by MÜNCH's symplasm hypothesis, we shall have to assume that the transport does not occur in the cytoplasm of the cells, but that in addition the cell-wall and the outer layer of the protoplasm will have to be passed. As the cell-wall is comparatively thin and the outer layer of the plasm allows the ions to pass, the ion transport will in principle take place in the same way as in a symplasm, but it will be greatly retarded by the diffusion from cell to cell.

The mechanism of the ion transport in the plasm is too hypothetical as yet to be treated extensively. In a previous publication (ARISZ 1944) we pointed out when discussing the transport in Drosera tentacles that the transport of ions in the cytoplasm through binding to the protoplasm is conceivable in two ways. It may be imagined 1. that *the ions bound to protoplasmic particles* are transported by the streaming protoplasm, so that we have to deal with *protoplasmic streaming*; 2. that the ions first bound to the outer layer of the protoplasm proceed to other plasmic-particles and from these again to others, etc., so that therefore *the ions are transported in the plasm, each time bound to other particles of protoplasm.*

The first hypothesis was also discussed by LUNDEGARDH in 1932. He says on p. 227: :"Durch solche Massenströmungen würde natürlich ein Durchtritt von gelösten Körpern auch in dem Fall stattfinden können, wenn die Diffusionspermeabilität sehr gering oder gleich Null ist. Wenn nämlich der gelöste Körper durch chemische Bindung oder Adsorption von den Partikeln der äusseren Grenzschicht des Kolloids aufgenommen wird, so kann es bei Konvektionsbewegung der Kolloidpartikeln doch durch die Schicht hindurchgehen. Dieser theoretisch denkbare Fall von Konvektionspermeabilität scheint bisher nicht berücksichtigt worden zu sein."

LUNDEGARDH points in this connection to the theoretically conceivable case that ions are taken up without permeating into the plasm (cf. LUNDEGARDH 1940 p. 263). As, however, interchange of ions also takes place at a low temperature, at which the protoplasmic streaming ceases, this possibility should in my opinion not be considered as an explanation of the passing of the surface layer.

A transport of substances bound to protoplasmic particles virtually resolves itself into HUGO DE VRIES' old theory on the influence of protoplasmic streaming on the transport. According to this the movement of the plasmic particles causes the transport of substances.

The second hypothesis that the ions go from one plasm-binding to another, has certain advantages, especially if the conception of a symplasm is correct. For in that case the ions can be spread over the whole symplasm in the same way. If, however, the symplasm-hypothesis is not correct, we shall have to assume that the ions get from one cell into the other by diffusion and in the case of polar transport by electric forces as well, by which each time both the outer layer of the protoplasm and the cell wall have to be passed.

Indeed there is not a very great difference between the two hypotheses, as probably the invisible transport of ions will bring about visible protoplasmic streaming, as VAN DEN HONERT (1932) has proved likely by means of model experiments. In that case protoplasmic streaming is not the cause of the transport of substance, but an attending phenomenon. FITTING's researches with Vallisneria leaves on chemodinesis of various substances would indicate that owing to the emersion of a leaf in a solution containing a chemodinetically working substance, the latter is taken up in the cells and causes a microscopically perceptible streaming in the protoplasm. When after some time the substance is equally distributed over the complex of cells, the protoplasm again settles down. This is not the place to point out the many points of correspondence between protoplasmic streaming and transport of substance, both in the transport of growth substance (BOTTELIER, DU BUY and OLSON, THIMANN and SWEENEY) and in the transport of salts (ARISZ, tentacles of Drosera and still unpublished researches on salt transport in Vallisneria). The theory of JENNY and OVERSTREET (1939 cf. fig. 2) on transport along

surface boundaries indicates a possibility, how transport of ions along surfaces can rapidly take place (cf. LUNDEGARDH 1940, p. 369). If the ions in the plasm are bound to substances like proteins, they may also interchange in a similar way between adjacent ion-binding areas. If, however, the ions are more firmly attached to the proteins, so that such

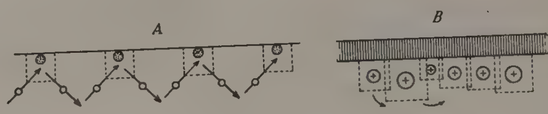

Fig. 2.

A. Schematic representation of diffusion of ions in colloids.
The dotted lines circumscribe the oscillation spaces of the adsorbed ions.
B. Scheme of the movement of a positive ion along a boundary.
The oscillation spaces of the adsorbed ions partly coincide, which makes a faster movement possible.
(From JENNY and OVERSTREET.)

an interchange cannot easily take place, we may be reminded of the short life of these proteins, as it appears from newer data on substances containing radio-active atoms, in which the ions are set free, when the protein is decomposed. May be rhythmical processes as JANSSEN imagines by the synthesis of substances in the organism, may act a part in these ion movements. The cause of these ion movements will have to be found in concentration differences which are caused either by a higher active concentration in the medium or by a lower active concentration in the regions of growth and synthesis in the plant. In § 8 this will be further discussed. Seeing the transfer of unequal amounts of cations and anions would cause an accumulation of electrical charges, a simultaneous transport of anions and cations in the protoplasm will as a rule have to be assumed. In how far in synthesis processes cations and anions can locally be used up in unequal quantities and the levelling of the difference in charges may take place at a different point in the plant, we cannot express an opinion on. BREAZEALE (1923) seems to have conceived something like this.

§ 6. *Permeability of tissues.*

It appears from the preceding discussion that in order to understand the transport of ions in tissues and particularly in the root, we must get away from considering the behaviour of separate cells. We must consider the parenchymatous tissue as one whole and therefore we speak of the permeability of a tissue in distinction of the permeability of a cell or of the plasm of one single cell. The phenomenon of the plasm-transmeability (see the definition on p. 25) is very material from the standpoint of *cell* physiology, because it defines what substances diffuse in the vacuole. From the standpoint of *tissue* physiology, however, this cellular phenomenon is less important. Here we are not concerned with

the introduction of substances into the vacuole, but with the permeability of the plasm in that sense that substances can pass the plasm of the whole tissue. The permeability of the tissue concerns the penetration into the protoplasm (intrability), the further transport through the plasm from cell to cell and the possible release of the substance. In this process the outer surface layer of the plasm is passed at any rate ,but this need not be the case with the tonoplast, so that the being taken up into the vacuole or not is of no consequence. Hence the fact that from experiments from which it appears that a tissue is permeable, we may at most draw conclusions as to the intrability of the cytoplasm, but not as to its transmeability, while it must be considered whether the substance may be transported along the cellwall without concerning the protoplasm. An instance, making this clear, has already been discussed viz. the permeating of concentrated salt solutions through the endodermis sheath of the root (cf. p. 4). Here we still wish to discuss K. PERIS' experiments (1936). She found that roots of Phaseolus multiflorus, which suck up water from a potetometer with a constant speed, take up less liquid from the moment when the water is replaced by a salt solution. It is comprehensible that the suction tension of the salt solution retards the uptake of water (BRIEGER 1928). After some time, however, the uptake of water increases without reaching its original strength. This increase in water absorption she explains by permeation of the salt solution into the cells of the root and she makes use of this phenomenon to compare the permeation of various salt solutions. By permeation she means the penetration of substances through the plasm into the vacuole. Let us assume that this phenomenon is indeed connected with the penetration of salts into the tissue of the root. [1]) It is then permitted to speak of plasm permeability, as only the penetration of the salts into the protoplasm (symplasm) in the direction of the woodvessels is concerned. If this takes place the resistance of the symplasm decreases and the absorption of fluid must increase. Only the cell wall and the plasmalemma need be passed. From these experiments, therefore, we may at most draw a conclusion about the *intrability* of the plasm and the ability of special substances to be transported through the symplasm of the root cells. On the *transmeability* of the plasm and the permeability of the tonoplast these experiments can certainly not give us any information. That is why a comparison of the results of these experiments with those of FITTING, BARLUND and HÖFLER. who investigated into the transmeability, is not permitted. Similar confusions are repeatedly found in literature (cf. p. 424).

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[1]) Here we may point to the possibility that as a result of decreased water supply, the suction tension in the leaf cells, which is the cause of water transport, increases. As for bleeding SABININ found a similar phenomenon, it is probable that the penetration of salts also acts a part.

§ 7. *Salt-transport in the root.*

In the preceding discussion the process of ion-transport has only been discussed in its simplest form, as it will proceed in a symplasm or in a complex of identical cells. We already discussed in § 1 that in the root two processes take place side by side: 1. an accumulation of substances into the vacuoles of the parenchymatous cells of the cortex and 2. a transport of substances to the central cylinder and secretion to the wood vessels. By the root hairs of the epidermal cells salts are absorbed from the environment. They must be transported through the plasm of the cortical cells to the central cylinder, but on their way there the plasm of the cortical cells may be able to secrete salts into the vacuoles of these cells. This does not preclude that salts also penetrate into the root through the walls of the parenchyma cells of the cortex. Together with the absorbed water they are transported in the intermicellar spaces of the cell walls (STRUGGER and ROUCHAL) and may penetrate into the plasm of the cortical cells. This renders it comprehensible that with a stronger water-absorption more salts are taken up, because in that case not only the whole surface of the epidermal cells, but also that of the cortical cells adjoins the external solution. Of course this is only material if the concentration in the external solution is very low, so that the strength of the absorption is also determined by the extension of the absorbing surface.

The salts which are transported in the cell walls and in the plasm of the cortical cells of the root, arrive through the endodermis in the central cylinder and are there given off to the wood vessels. The concentration in these vessels can become considerably higher than in the medium (HOAGLAND and BROYER). Together with the water present in the vessels the salts are transported to other parts of the plant and again taken up into the plasm and the vacuole by the living cells of leaves and branches. Part of the salts is absorbed in the growing parts by the synthesis of protoplasmic substances, another part is being secerned in the vacuoles the remainder would be transported to the basal parts of the plant through the sieve vessels. According to MASON and MASKELL (1931) nitrogen, phosphor and probably also potassium and some other ash-constituents would take part in this transport, while calcium remains in the leaves and is not transported in the phloem.

The process in which salts are absorbed from the medium and excreted to the wood vessels, requires a further discussion. CRAFTS and BROYER gave an interesting explanation of this. According to them the external conditions, especially the oxygen supply would be for the cortical cells different from what it is for the cells lying inside the central cylinder, because the former tissue is well-supplied with oxygen through air canals, whereas in the central cylinder the cells fit together without intercellular spaces and are consequently badly provided with oxygen. Under these circumstances the cortical cells would take up salts actively; but the cells

of the central cylinder release salts. It is quite possible that this theory of CRAFT and BROYER's is based on a correct thought. It assumes, however, an active salt accumulation by the cortical cells in the protoplasm. According to STEWARD, HOAGLAND and others, the active accumulation in the cortical cells is based chiefly on the accumulation in the vacuole and this is of no consequence for a transport to the central cylinder (cf. § 4).

The salt transport from medium to woodvessels can't be a simple diffusion process, because the concentrations of cations and anions in the xylem vessels may be higher than in the medium, so that the uptake and transport of ions to the wood may occur contrary to a concentration gradient. This cannot but imply that we have to deal with an active mechanism. This mechanism, may be, as we saw, the accumulation mechanism of the cortical cells, but there is also another possibility which will be discussed here.

In this connection we may remind of the fact that in experiments on the influence of the environment on the composition of the bleeding sap, a similar result was obtained, viz. that the ions in the bleeding sap may be present in a higher concentration than in the environment. LAINE (1934) found that here the same connection exists between the concentration of the bleeding sap and that of the environment as with an adsorption process between the amount of adsorbed substance and the concentration of the environment, so that FREUNDLICH's formula holds good, $s = k\,c^{\frac{1}{n}}$ in which s = concentration of the bleeding sap and k and n represent constants which are different in the case of potassium, calcium and manganese. This points to the fact that the process of bleeding, by which substances from the environment are absorbed and transported to the xylem conforms to the same law as the absorption by the root and the leaves. In § 3 we ascribed this phenomenon to the adsorption-binding of the ions from the environment to the protoplasm and the removal of the adsorbed ions at a constant speed. With the secretion of substances into the vacuole the cause of the transport of ions is to be found in an active process that secretes the ions into the vacuole. Here a similar active process may assert itself which removes the ions from the surface layer and which is the cause of their transport in the symplasm of the cortical cells. As a result new ions are continuously adsorbed from the environment, next transported and finally given off to the woodvessels. The concentration of the ions in the woodvessels can then rise above that in the environment. The situation therefore is such that in the symplasm and the cell walls of the central cylinder a higher salt concentration may prevail than in that of the surrounding tissue.

Now it is known from anatomical data that on the boundary line of cortex and central cylinder the endodermis sheath is found. This can only allow the salts to pass through the plasm of the cells (DE RUFZ DE LAVISON 1911). The bands of CASPARY prevent a flow of the salt ions

from the central cylinder through the walls of the endodermis cells to the cortex and environment, so that here only transport of ions through the plasm of the endodermal cells can take place. It is therefore obvious to look in the endodermis for the cause of the active transport of ions, though it is not excluded that other cells in the central cylinder have the same function. If this conception should be correct, the cells of the endodermis would have a secretory function (cf. URSPRUNG 1925, GUTTENBERG). There are several data that indicate that this function varies specifically for different ions. Some ions are easily transported between environment and wood vessels, others less easily. WIERSUM's experiments, which have been made in this laboratory and have not yet been fully published, show this. WIERSUM (1944) traced bij roots of Vicia Faba how salts brought into the woodvessels can permeate through the central cylinder and the cortex to the environment. He found that calcium-ions pass well and potassium-ions less easily. These are experiments in which the permeability of the root tissue was examined, and from which it appears that calcium-ions can be easily transported through the root tissue in a radial direction. In WIERSUM's experiments the uptake of water proceeded in a direction opposite to that of the salt transport. The transport, therefore, need not be a passive carrying along by a watercurrent, but may very well be based on diffusion and ion movements in the cytoplasm.

SCHMIDT (1936) found that the uptake of ions in Sanchezia nobilis is accelerated by transpiration. This obtains for calcium, magnesium, nitrate and phosphate, but not for potassium. BÖTTICHER and BEHLING found for maize that transpiration but slightly accelerates the uptake of potassium and phosphate, but strongly accelerates that of calcium. All these experiments show that calcium can fairly easily be transported through the system: environment-cortex-endodermis-centralcylinder-woodvessels, but that potassium behaves differently. It may be taken into consideration, whether the different behaviour of these ions is dependent on the more or less active transport of these ions.

Though not in a direct way concerned with the permeability of salts, yet it is worth while pointing out in this connection that also organic substances behave in the same way. Both PERIS and WIERSUM state that *sucrose* permeates fairly easily through root tissue. This result shows that *the surface layer of the cytoplasm is permeable for sucrose*. Evidently we have methods in hand here to ascertain the *intrability* of the plasm for various substances.

Of course it need not be added that the permeability of the tonoplast for sucrose is a problem in itself. In the case of sugar it may be expected that the outerlayer is permeable, the tonoplast is not. The plasm therefore is intrable for sucrose. WEEVERS already defended this conception in 1931 (cf. also NATHANSON and BENECKE and JOST I. p. 32, 1924) and could explain both the phenomena in plasmolysis through a sucrose-solution and the formation of starch in the cells of leaves that floated on a sugar

solution. If sugar is found in the vacuole, this must be a result of metabolic processes, in which the sugars are accumulated or perhaps more exactly secerned into the vacuole, so that an accumulation may be brought about there.

§ 8. *Schematic representation of the processes for the uptake of electrolytes by the plant.*

In the preceding sections the different sides of the problem, how ions are taken up, have been discussed. We shall now proceed to treating a scheme on the uptake of salts and other electrolytes by the plant, which gives a summary of the insight obtained and consequently links up with the conceptions of other investigators.

Such a schematic representation offers the advantage that various points must be accurately formulated, in doing which it may appear how far we are still removed from a correct insight into these intricate processes.

The process of the uptake of ions may be divided into various phases. The first two phases are the uptake of the ions into the outer layer of the protoplasm, their binding at certain points and the movement of the ions in the protoplasm or in the symplast. If the temperature in which the organism is, is low or the vital processes are inhibited in another way, only an interchange of ions is possible. This occurs when the plant is transferred to an environment of a different composition. If, however, the plant is active, a third phase appears, in which ions are removed from the plasm by several causes. The local consumption of ions causes a disturbance of equilibrium in the symplasm, of which a movement of ions is the result and which ultimately causes an uptake of fresh ions from the environment. In this case an equal amount of cations and anions will have to be removed or if either of them preponderates, ions from the plant will have to be given off to the environment at the same time.

Removal of ions from the symplasm may take place in *three* different ways, viz. by *consumption of ions through the synthesis of new substances, through secretion of ions to the vacuole,* and *through actively secerning cells.* For the development of these processes in the living organism a protoplasm functioning normally is required. Besides both synthesis and secretion are dependent on respiration, because both are processes using up energy. As a result these processes will no more proceed normally in case of withdrawal of oxygen.

After all it is of little consequence whether we identify the active uptake of substances with the whole process of the uptake of ions from the outer layer to their consumption, or whether we will only give this name to part of this process, viz. the secretion to the vacuole. We shall speak of active uptake, when ions are withdrawn from the medium and they are fixed in either of the above ways by secretion to the vacuole or to other places or by using up by synthesis.

It has been indicated in the scheme (fig. 3) that between medium M and vacuole V the cytoplasm is found with its surface layers plasmalemma P_1 and tonoplast P_2. With the active uptake by a cell adjoining the outer solution, the ions will have to pass these layers in order to be taken up into the vacuole. The scheme shows that the accumulation in the vacuole

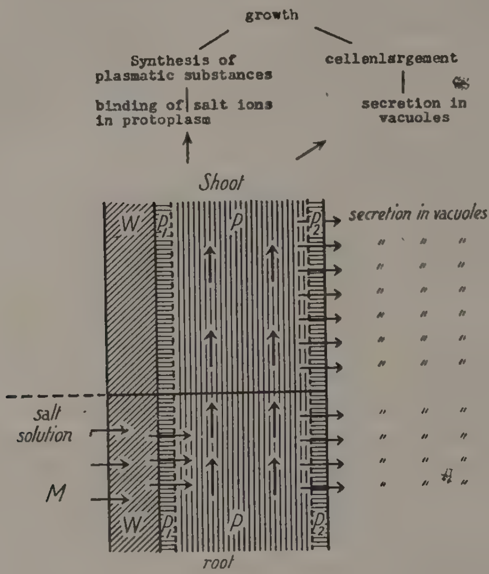

Fig. 3.

The uptake and transport of salts in parenchymatic tissue of root and shoot.
W = cell wall.
P = symplasm.
P_1 = plasmalemma.
P_2 = tonoplast.
M = outersolution.

may take place by secretion of ions by the cytoplasm into the cell sap. In this process the tonoplast does not allow these ions to pass through diffusion. Ions being in the protoplasm may be moved in the protoplasm without needing to pass plasmalemma or tonoplast. This is expressed by the conception symplast.

In the symplast a movement of ions can take place and in every cell-vacuole ions can be accumulated by a mechanism lying in the adjoining protoplasm.

In addition the scheme shows that in growing parts ions are fixed. Growth is dual: growth of plasm owing to synthesis of cell substances, when ions are fixed, and besides growth in volume by stretching of the cellwall.

The process of cell enlargement is based on turgor, active intussusception, growth substance and cell building material (cf. p. 14). The

tendency to maintain the turgor pressure and the taking up of water by the vacuole must be the result of simultaneous secretion of osmotically working substances in the vacuole, which causes water to be taken up. I can not agree with FREY WYSSLING who postulates an active uptake of water. As both the tonoplast and the plasmalemma are permeable to water this process would have no influence at all. I consider the active uptake of water, which has been demonstrated by the experiments of Miss REINDERS in this laboratory as a consequence of the active secretion of osmotic substances into the vacuole. Active secretion of substances into the vacuole is therefore conditional for the growth of the cell.

Both for synthesis and secretion ions are withdrawn from the symplast. By synthesis of amino-acids and proteins nitrogen in the form of NH_4 or NO_3 will be withdrawn from the symplast and S- and P-ions can also be fixed. The colloidal substances formed will moreover adsorb ions such as K, Mg, Ca, etc.

In growing parts therefore anorganic ions are absorbed directly and indirectly through synthesis, while besides a continuous secretion to the vacuoles takes place. The influence of the endodermis on the transport of the salts in the root has not been shown in this scheme (cf. § 7). The scheme only refers to parenchymatous tissue in general.

Summary.

An analysis is given of the uptake of salts by plants. This is a general physiological phenomenon common as well with unicellular organisms as with higher plants. Besides there are organs which have the special function of absorption and transport e.g. the tentacles of Drosera.

The root takes up salts from the surrounding solution or from the soil and moves them to the xylem-vessels and to the young growing cells at its apex. Moreover the cortical cells are able to accumulate salts in their vacuoles. This process is characteristic for most living cells, it requires energy that is produced by aerobic respiration.

The problem is considered whether the boundaries of the protoplast, the plasmalemma and the tonoplast are pervious to salts. Experiments with plasmolysis seem to prove this. Arguments are given that under normal conditions the plasmalemma is permeable to ions but that the tonoplast behaves differently and may be impermeable to certain ions. The data about the permeability of the boundary layers of the protoplasm are insufficient to give a definite opinion on this point.

The consequence of the impermeability of the tonoplast for certain substances or ions is that there must be an active secretion process in order to accumulate them into the vacuole.

The opinion is given that the ions in the peripheral layer of the protoplasm are bound to plasmatic substances and that there exists an adsorption-equilibrium between the ions in this layer and in the surrounding solution. The consequence of a change in the composition or

strength of this solution is a change not only in the peripheral layer of the protoplasm but also in the whole symplasm of the communicating cells.

If ions bound to protoplasmic particles are removed somewhere in the symplasm by local consumption, there results a transport of ions and new ions from the medium wil be bound. The strength of this ion-transfer is limited by the adsorption equilibrium at the surface layer.

The movement of the ions in the plasm is a submicroscopical process, which becomes perceptible in the microscopically visible protoplasmic streaming. This explains the parallelism between both processes, that has also been found with the transport of growth-substance.

The cause of the removal of ions from the symplasm may be

1. their use in synthetizing new compounds in the protoplasm and the binding of ions by the newly formed substances;
2. the secretion of ions out of the protoplasm to the vacuole;
3. the secretion of ions out of the protoplasm to the exterior e.g. to the xylem-vessels.

An accumulation of salt-ions in plant cells may be caused by different processes. It may be an accumulation in the cell-wall, in the protoplasm or in the vacuole. The ion-content of the protoplasm wil probably depend on Donnan equilibria and on the diffusion effect studied by Teorell. The accumulation of ions into the vacuole must be considered as an active secretion-process by the contiguous protoplasm, if the tonoplast is impermeable to these ions.

The above-exposed theory of salt uptake is in harmony with our knowledge about the salt content of the plant. The uptake of ions depends on the constitution of the surrounding solution though by specific adsorption by the protoplasm the ions can be absorbed in other ratio than they are present in the medium. The ions are transported for a great deal to places where they are used for the building or enlarging of young cells. As a consequence the quantity of ions present in the plant depends as well on the composition of the outer solution as on the specific properties of the various tissues.

LITERATURE.

Papers published in the allied countries after May 1940 could not be consulted. Only a small number of the titles mentioned in the text will be given here. The rest can be found easily in the well known works on permeability.

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Applied Mechanics. — *The generalized buckling problem of the circular ring.* By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of November 24, 1945.)

1. *Introduction.* It is well known, that a circular ring, subjected to a uniform radial pressure q per unit of circumferential length is apt to buckle under the action of one of the so-called "critical" loads $q = (n^2 - 1)EI/r^3$ (n integer and $\geqq 2$), EI representing the flexural rigidity of the ring and r its radius. This case of buckling is analogous to the buckling of a straight rod under the action of two compressive forces, as far as the cross-sections of both ring and rod are loaded by a normal force of constant magnitude. If a straight rod is loaded by a prescribed system of axial forces, so that the normal force of the cross-section varies with its coordinate, proportional increase of the loadsystem, say to the multiple λ, leads as well to critical buckling loads. The (positive or negative) value of the factor of magnification λ can best be found by a method of iteration [1]. It is obvious, that for the circular ring the analogous problem exists, if only it is subjected to an external loadsystem such that in every crossection of the ring the bending moment M and the shearing force D are zero, whereas the normal force of the section varies with its coordinate. Evidently the first of these conditions will not be fulfilled if the ring is loaded by an arbitrary system, of radial and tangential forces; but is will be shown in section 2 that every loadsystem can be split up into two components A and B, the first of which will be called the "compressive" system, because it is characterised by $M = D = 0$, whereas the second one will be called the "bending" system, characterized as it is by $N = 0$.

The first system A, if suitably magnified, leads to the generalized buckling problem of the circular ring, which will be treated in this paper, but it is seen at once, that an arbitrary loadsystem, consisting of both components A and B, gives rise to a problem, which again is analogous to a well-known problem, viz. the straight rod subjected to axial thrust and transverse bending loads. Just as well as with the straight rod the axial forces tend to increase the deflections caused by the transverse loads, the A loadsystem of the circular ring will affect the deflections due to the B-system. In a subsequent paper this latter question will be treated in connection with a particular problem, which led the authors to the present investigation, and which itself will be treated in a third communication.

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[1]) Comp. f. i. C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, Chapt. VII, 7, 509 (Springer 1939).

2. *The A and B-system.* If a ring is subjected to radial and tangential loads q and t per unit of circumferential length the equilibrium of a ringelement requires

$$\left.\begin{array}{lll} N\,d\varphi - qrd\varphi - dD = 0 & & N - D' = qr \\ dN + Dd\varphi + trd\varphi = 0 & \text{resp.:} & N' + D = -tr \\ dM + rdN + tr^2\,d\varphi = 0 & & M' + rN' = -tr^2 \end{array}\right\} \quad (1)$$

(φ denoting the angular coordinate of the ringelement under consideration). The requirement $D = M = 0$ leads to

$$N = qr \quad , \quad N' = -tr \quad (2)$$

from which it follows, that a A-loadsystem is characterized by

$$t = -q' \quad (3)$$

At the other hand the condition $N = 0$ requires

$$-D' = qr \qquad D = -tr \qquad M' = -tr^2 \quad (4)$$

from which it is seen, that a B-loadsystem is characterized by

$$q = t' \quad (5)$$ [2]

It can easily be shown that any arbitrary equilibrium loadsystem (q, t) can be decomposed in a unique way into a A-system (q^*, t^*) and a B-system (q^{**}, t^{**}). Let all quantities be expanded into their Fourier-series, so that:

$$q = a_0 + \sum_1^\infty a_k \cos k\varphi + \sum_1^\infty b_k \sin k\varphi,\ t = c + \sum_1^\infty c_k \cos k\varphi + \sum_1^\infty d_k \sin k\varphi \quad (6)$$

$$q^* = a_0^* + \sum_1^\infty a_k^* \cos k\varphi + \sum_1^\infty b_k^* \sin k\varphi,\ t^* = c_0^* + \sum_1^\infty c_k^* \cos k\varphi + \sum_1^\infty d_k^* \sin k\varphi \quad (7)$$

$$q^{**} = a_0^{**} + \sum_1^\infty a_k^{**} \cos k\varphi + \sum_1^\infty b_k^{**} \sin k\varphi,\ t^{**} = c_0^{**} + \sum_1^\infty c_k^{**} \cos k\varphi + \sum_1^\infty d_k^{**} \sin k\varphi(\quad (8)$$

then it must be remarked beforehand, that in consequence of the equilibrium-conditions

$$\int_0^{2\pi} (q \sin\varphi + t \cos\varphi)\, r\, d\varphi = 0,\ \int_0^{2\pi} (q \cos\varphi - t \sin\varphi)\, rd = 0,\ \int_0^{2\pi} tr^2\, d\varphi = 0 \quad (9)$$

the following relations hold

$$b_1 + c_1 = 0 \quad a_1 - d_1 = 0 \quad c_0 = 0. \quad (10)$$

Analogously the conditions

$$b_1^* + c_1^* = 0 \qquad a_1^* - d_1^* = 0 \qquad c_0^* = 0 \quad (11)$$

$$b_1^{**} + c_1^{**} = 0 \qquad a_1^{**} - d_1^{**} = 0 \qquad c_0^{**} = 0 \quad (12)$$

must be fulfilled.

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[2]) Strictly spoken it must be proved that inversely (3) leads to $D = M = 0$, and (4) to $N = 0$. This may be left to the reader.

Furthermore we have, in consequence of $(q, t) \equiv (q^*, t^*) + (q^{**}, t^{**})$

$$a_0^* + a_0^{**} = a_0 \qquad a_k^* + a_k^{**} = a_k \qquad b_k^* + b_k^{**} = b_k \quad . \quad . \quad (13)$$

$$c_0^* + c_0^{**} = c_0 \qquad c_k^* + c_k^{**} = c_k \qquad d_k^* + d_k^{**} = d_k \quad . \quad . \quad (14)$$

in consequence of $t^* = -q^*$:

$$c_0^* = 0 \qquad c_k^* = -k\, b_k^* \qquad d_k^* = k\, a_k^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in consequence of $q^{**} = t^{**}$:

$$a_0^{**} = 0 \qquad -k\, c_k^{**} = b_k^{**} \qquad k\, d_k^{**} = a_k^{**}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From these equations we deduce, having due regard to (10), (11) and (12)

$$\left.\begin{array}{ll}
\multicolumn{2}{l}{a_0^* = a_0,\ a_0^{**} = c_0^* = c_0^{**} = 0} \\
\multicolumn{2}{l}{a_1^* = a_1^{**} = \tfrac{1}{2} a_1,\ b_1^* = b_1^{**} = \tfrac{1}{2} b_1,\ c_1^* = c_1^{**} = -\tfrac{1}{2} b_1,\ d_1^* = d_1^{**} = \tfrac{1}{2} a_1} \\
a_k^* = -\dfrac{1}{k^2-1} a_k + \dfrac{k}{k^2-1} d_k & b_k^* = -\dfrac{1}{k^2-1} b_k + \dfrac{k}{k^2-1} c_k \\
a_k^{**} = \dfrac{k^2}{k^2-1} a_k - \dfrac{k}{k^2-1} d_k & b_k^{**} = \dfrac{k^2}{k^2-1} b_k + \dfrac{k}{k^2-1} c_k \\
d_k^* = -\dfrac{k}{k^2-1} a_k + \dfrac{k^2}{k^2-1} d_k & c_k^* = \dfrac{k}{k^2-1} b_k + \dfrac{k^2}{k^2-1} c_k \\
d_k^{**} = \dfrac{k}{k^2-1} a_k - \dfrac{1}{k^2-1} d_k & c_k^{**} = -\dfrac{k}{k^2-1} b_k - \dfrac{1}{k^2-1} c_k
\end{array}\right\} \quad (17)$$

Therefore the A and B components of the loadsystem (q, t) are represented by

$$\left.\begin{array}{l}
q^* = a_0 + \tfrac{1}{2} a_1 \cos\varphi + \sum\limits_2^\infty \left[\dfrac{-1}{k^2-1} a_k + \dfrac{k}{k^2-1} d_k\right] \cos k\varphi + \tfrac{1}{2} b_1 \sin\varphi + \\
\qquad\qquad + \sum\limits_2^\infty \left[-\dfrac{1}{k^2-1} b_k - \dfrac{k}{k^2-1} c_k\right] \sin k\varphi \\
t^* = \tfrac{1}{2} b_1 \cos\varphi + \sum\limits_2^\infty \left[\dfrac{k}{k^2-1} b_k + \dfrac{k^2}{k^2-1} c_k\right] \cos k\varphi + \tfrac{1}{2} a_1 \sin\varphi + \\
\qquad\qquad + \sum\limits_2^\infty \left[-\dfrac{k}{k^2-1} a_k + \dfrac{k^2}{k^2-1} d_k\right] \sin k\varphi
\end{array}\right\} \quad (18a)$$

$$\left.\begin{array}{l}
q^{**} = \tfrac{1}{2} a_1 \cos\varphi + \sum\limits_2^\infty \left[\dfrac{k^2}{k^2-1} a_k - \dfrac{k}{k^2-1} d_k\right] \cos k\varphi + \tfrac{1}{2} b_1 \sin\varphi + \\
\qquad\qquad + \sum\limits_2^\infty \left[\dfrac{k^2}{k^2-1} b_k + \dfrac{k}{k^2-1} c_k\right] \sin k\varphi \\
t^{**} = -\tfrac{1}{2} b_1 \cos\varphi + \sum\limits_2^\infty \left[-\dfrac{k}{k^2-1} b_k - \dfrac{1}{k^2-1} c_k\right] \cos k\varphi + \tfrac{1}{2} a_1 \sin\varphi + \\
\qquad\qquad + \sum\limits_2^\infty \left[\dfrac{k}{k^2-1} a_k - \dfrac{1}{k^2-1} d_k\right] \sin k\varphi
\end{array}\right\} \quad (18b)$$

In the following sections attention will only be paid to A-load systems.

3. *The differential equation of the buckling ring.* The differential equation of a bent circular ring, the central line of which does not change its length is represented by

$$\frac{1}{\varrho}-\frac{1}{r}=\frac{Mr^2}{EI} \quad \text{. . . . .} \quad (19)$$

($1/r$ = curvature of the unbent ring, $1/\varrho$ = curvature of the bent ring; M = bending moment, positive if it increases the curvature of the ring; EI = flexural rigidity of the ring). From geometrical deductions it follows that

$$\frac{1}{\varrho}-\frac{1}{r}=-\frac{u''+u}{r^2} \quad \text{. . . . .} \quad (20)$$

u representing the increase of the radius vector r. From eqs. (1) it follows, by eliminating N and D:

$$M''+M'=-(q'+t)\,r^2 \quad \text{. . . . .} \quad (21)$$

and from (21) and (19) we find

$$(u''+u)'''+(u''+u)'=\frac{(q'+t)\,r^4}{EI} \quad \text{. . . . . .} \quad (22)$$

This equation can be applied to a buckling ring subjected to a critical A-loadsystem (q^*, t^*) if only q and t be replaced by those supplementary loadcomponents to which — by the deformation of the ringelement — the prevailing external and internal loads $q^* r d\varphi$, $t^* r d\varphi$, N and $N+dN$ give rise. Two different cases must be distinguished, according as the external loads q^* and t^* keep their directions with respect to the ring-element or with respect to a fixed system of coordinates. In this paper we confine ourselves to the first one, so that q has to be replaced by $-N\left(\frac{1}{\varrho}-\frac{1}{r}\right)=N(u''+u)/r^2$ (comp. 20) and t by zero.

Consequently eq. (22) changes into:

$$(u''+u)'''+(u''+u')=\frac{[N(u''+u)]'\,r^2}{EI} \quad \text{. . . . .} \quad (23)$$

If we denote $u''+u$ by U, eq. (23) can be replaced by

$$U''+U=\frac{Nr^2}{EI}\,U+C \quad \text{. . . . .} \quad (24)$$

where C represents a constant of integration to be found in the course of our investigation.

Both quantities N and U are periodic functions of φ with the period 2π,

so that they may be expanded into Fourier-series:

$$\left.\begin{aligned} N = \lambda N_0 = \lambda \left[A_0 + \sum_{k=1}^{\infty} A_k \cos k\varphi + \sum_{k=1}^{\infty} B_k \sin k\varphi \right] \\ U = a_0 + \sum_{l=1}^{\infty} a_l \cos l\varphi + \sum_{l=1}^{\infty} b \sin l\varphi \end{aligned}\right\} \quad . \; . \quad (25)$$

These expressions, however, are subject to some distinct restrictions, which must be discussed beforehand.

Firstly it must be remembered that we have to deal with a closed ring, so that — if the ring was cut at say $\varphi = 0$, and if the disturbed internal forces were restored — neither relative displacements nor relative rotation of the opposite ends of the ring would occur. These conditions are analytically expressed by the equations

$$\int_0^{2\pi} \frac{M}{EI} r d\varphi = 0, \quad \int_0^{2\pi} \frac{Mr \sin\varphi}{EI} r\, d\varphi = 0, \quad \int_0^{2\pi} \frac{Mr\,(1-\cos\varphi)}{EI} rd\varphi = 0 \quad . \; . \quad (26)$$

relating to well-known properties of the so-called "reduced" bending moment M/EI. As, according to (19) M/EI is proportional to $u'' + u = U$, we find

$$\int_0^{2\pi} U\, d\varphi = 0, \qquad \int_0^{2\pi} U \sin\varphi\, d\varphi = 0, \qquad \int U \cos\varphi\, d\varphi = 0 \quad . \; . \quad (27)$$

from which it follows that

$$a_0 = a_1 = b_1 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

All further restrictions (relating to N) follow from the fact, that (24) must *identically* be fulfilled if N and U are replaced by the expressions (25). To control this required identity the product NU, occurring at the right hand side of (24), evidently must be written in terms of cos and sin of multiples of φ. In doing so it is seen at once, that no terms $A_1 \cos\varphi$ and $B_1 \sin\varphi$ in N are admitted if U contains the terms $a_2 \cos 2\varphi$ and $b_2 \sin 2\varphi$; otherwise the product NU would give rise to terms $\cos\varphi$ and $\sin\varphi$ in the right hand side of (24), which in virtue of (28) are missing in the lefthand side. The presence of $A_1 \cos\varphi$ and $B_1 \sin\varphi$ would require the disappearance of $a_2 \cos 2\varphi$ and $b_2 \sin 2\varphi$ in U, so that the Fourier-series for U should have to start with $a_3 \cos 3\varphi$ and $b_3 \sin 3\varphi$. However, the product of these terms with $A_1 \cos\varphi$ and $B_1 \sin\varphi$ would in the righthand side of (24) now produce terms with $\cos 2\varphi$ and $\sin 2\varphi$ which, in consequence of the preceding remark, are absent in the lefthand side. Proceeding in this way *all* terms of U must disappear, and therefore we conclude, that no equilibrium position of the ring, different from the circular one, exists if A_1 and B_1 are different from zero; any departure of the circular shape sets the ring into motion.

Moreover all multiples of φ_1 occurring in the remaining terms of N must have a factor, different from one, in common. The statement is proved by showing first, that the presence of two terms cos $k_1\varphi$ and cos $k_2\varphi$, k_1 and k_2 having no factor in common, leads to an absurdity. It is seen at once, that in such a case U is deprived from its terms cos $(k_1 \pm 1)\varphi$, sin $(k_1 \pm 1)\varphi$, cos $(k_2 \pm 1)\varphi$, sin $(k_2 \pm 1)\varphi$, because otherwise the righthand side of (24) would contain the terms cos φ and sin φ, which do not occur in the left hand side. But then terms of the type cos $(2k_1 \pm 1)\varphi$, sin $(2k_1 \pm 1)\varphi$, cos $(2k_2 \pm 1)\varphi$, sin $(2k_2 \pm 1)\varphi$ are as well forbidden terms for U; if such terms would exist, the righthand side of (24) would contain terms cos $(k_1 \pm 1)$ a.s.o. which — after the preceding statement — do not occur in the lefthand side. Proceeding in this way, we find that all terms cos $(\alpha k_1 \pm 1)\varphi$, sin $(\alpha k_1 \pm 1)\varphi$, cos $(\beta k_2 \pm 1)\varphi$, sin $(\beta k_2 \pm 1)\varphi$, (α and β representing arbitrary pos. or neg. integers) are forbidden terms for U. A similar reasoning leads to the conclusion firstly, that terms of the type cos $(\alpha k_1 \pm k_2 \pm 1)\varphi$, sin $(\alpha k_1 \pm k_2 \pm 1)\varphi$ are inadmissable, then that all terms cos $(\alpha k_1 \pm \beta k_2 \pm 1)\varphi$, sin $(\alpha k_1 \pm \beta k_2 \pm 1)\varphi$ must be excluded in U (α and β representing arbitrary pos. or neg. integers). But this means that *all* terms in U have to be excluded because of the fact that any integer may be written as $\alpha k_1 + \beta k_2$, provided only that k_1 and k_2 have no factor in common. Again, if N should contain the terms cos $k_1\varphi$, cos $k_2\varphi$, cos $k_3\varphi_1$ — k_1, k_2, k_3 having no factor in common — all terms cos resp. sin $(\alpha k_1 + \beta k_2 + \gamma k_3 \pm 1)\varphi$ (α, β, γ pos. or neg. integers) would be excluded from U, but this again would mean that all terms of U should vanish; a.s.o. Our final conclusion with respect to the possibility of buckling of the ring therefore is:

1°. All multiples of φ in the Fourier series of N must have a factor in common.

2°. If the greatest factor in common of all these multiples is called p, the terms to be excluded from U are a_0, $a_1 \cos\varphi$, $b_1 \sin\varphi$, $a_{\alpha p \pm 1}$ cos $(\alpha p \pm 1)\varphi$, $b_{\alpha p \pm 1}$ sin $(\alpha p \pm 1)\varphi$, α representing any integer pos. number $\neq 0$.

One way in which we could now pursue the solution of eq. (24) would consist in substituting the expressions (25) — liable to restrictions laid upon them — into (24), and by identifying the corresponding terms of both sides of this equation; this would lead to an infinite system of recurrent relations between the coefficients a_l and b_l. Though in fact we do not intend to follow this way, one useful remark, to which this method would lead us, must be made, viz. that the system of relations just mentioned breaks up into a number of minor systems, each of which relates to a distinct class of coefficients a_l, b_l, l being defined by

$$l \equiv \pm q, \bmod p \qquad \left.\begin{array}{ll} q = 0, 2, 3 \ldots \frac{p-1}{2}, & p \text{ odd} \\ q = 0, 2, 3 \ldots \frac{p}{2}, & p \text{ even} \end{array}\right\} \quad . \quad . \quad . \quad (29)$$

(If $q = 0$, the value $l = 0$ obviously must be excluded). With any prescribed p our buckling problem therefore is split up into $\frac{p-1}{2}$ or $\frac{p}{2}$ cases in every one of which U is composed exclusively of terms relating to one of the congruences (29).

We conclude this section by a remark with respect to the constant of integration C, occurring in eq. (24). If the product NU in the righthand-side of this equation happens to miss a constant term after being written as an ordinary Fourier-series, then C must be suppressed; if not so C serves to annul this term.

4. *The integral equation of the problem.* In the next sections up from sect. 5, eq. (24) will be solved in an iterative way. The justification of this method, however, makes it desirable to express our problem in terms of an integral equation, which therefore will be deduced first. To this end we apply to the ring an equilibrium loadsystem consisting of a concentrated force P of unit magnitude, acting at the fixed point (ψ) and a complementary continuous radial load q_φ, which at any point (φ) of the ring amounts to $q_\varphi = -P \cos(\varphi - \psi)/\pi r$. The bending moment at the point φ of the ring, due to this loadsystem is called $K(\varphi, \psi)$. It has been stated in sect. 3 that the deformation of the buckling ring must be maintained by the continuous radial load $\lambda N_0(\psi)\, U(\psi)/r^2$, where $\lambda N_0(\psi)$ stands for the normal force in the cross-section (ψ) and $-U(\psi)/r^2$ for the local change in curvature. If every infinitesimal part $\lambda N_0(\psi)\, U(\psi)\, r d\psi/r^2$ of this load (acting on the ring element $r d\psi$) is supplemented by a continuous load all over the ring of the specific amount $-\lambda N_0(\psi)\, r d\psi/r^2 - \cos(\varphi - \psi)/\pi r$, then the total bending moment M_φ at the angle φ amounts — in accordance to the just given definition of $K(\varphi, \psi)$ — to

$$M_\varphi = \int_0^{2\pi} \frac{\lambda N_0(\psi)\, U(\psi)\, K(\varphi, \psi)}{r}\, d\psi \quad . \quad . \quad . \quad . \quad . \quad (30)$$

whereas at the other hand (comp. (20))

$$M_\varphi = -\frac{E I\, U(\psi)}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Consequently U_φ satisfies the integral equation

$$U_\varphi = -\int_0^{2\pi} \frac{\lambda r\, N_0(\psi)\, U(\psi)\, K(\varphi, \psi)}{E I}\, d\psi \quad . \quad . \quad . \quad . \quad . \quad (32)$$

which also may be written as follows:

$$\left.\begin{aligned}\left[\sqrt{\frac{N_0(\varphi)\, r}{E I}}\, U(\varphi)\right] = -\lambda \int_0^{2\pi} \left[\sqrt{\frac{N_0(\psi) \cdot r}{E I}}\, U(\psi)\right] \left[\sqrt{\frac{N_0(\psi)\, r}{E I}}\, .\right. \\ \left. .\sqrt{\frac{N_0(\psi) \cdot r}{E I}}\, . K(\varphi, \psi)\right] d\psi\end{aligned}\right\} . \quad (33)$$

The kernel $\sqrt{\frac{N_0(\varphi)\cdot r}{EI}}\sqrt{\frac{N_0(\psi)\cdot r}{EI}}\;K(\varphi,\psi)$ of this equation is symmetrical with respect to φ and ψ in consequence of the fact that $K(\varphi,\psi)$ — by its mechanical meaning — is symmetrical with respect to its arguments. Therefore all general theorems, relating to homogeneous integral equations with a symmetrical kernel can be applied to this special equation. In particular it may be remembered: 1°. that there exists an infinity of characteristic numbers λ for which eq. (32) is satisfied in general by one and exceptionally by more corresponding characteristic functions U; 2°. that any two characteristic functions U_1 and U_2 corresponding to different characteristic number λ_1 and λ_2 are orthogonal in that $\int N_0(\varphi)\,U_1(\varphi)$ $U_2(\varphi)d\varphi=0$; 3°. that any arbitrary function $V(\varphi)$ can be expanded into a series of the characteristic functions U.

It must be emphasized that our deductions only have a bearing on our proper problem if the infinite collection of introduced supplementary continuous loads $-\lambda N_0(\psi)U(\psi)\cos(\varphi-\psi)d\psi/r^2$ does influence the ring nowhere. The effect of these loads at a fixed angle φ is represented by

$$-\int_0^{2\pi}\frac{\lambda N_0(\psi)\,U(\psi)\cos(\varphi-\psi)}{r}\,d\psi=-\lambda\cos\varphi\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\cos\psi}{r}\,d\psi-$$
$$-\lambda\sin\varphi\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\sin\psi}{r}\,d\psi$$

which really is zero in virtue of the equilibrium-equations

$$\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\cos\psi}{r}\,d\psi=0,\quad\int_0^{2\pi}\frac{N_0(\psi)\,U(\psi)\sin\psi}{r}\,d\psi=0$$

of the buckling ring.

Yet the problems represented by the eqs. (24) and (32) are not identical, for eq. (24) restricts itself to those loads NU/r^2 which are in equilibrium whereas eq. (32) equally refers to loads NU/r^2, which — not in equilibrium themselves — are balanced by a suitable load $a\cos\varphi+b\sin\varphi$. The reason of this discrepancy obviously is due to the fact, that in section 3 in the expression for N (25) the terms $\cos\varphi$ and $\sin\varphi$ and in the expression U (25) the terms $\cos l\varphi$ and $\sin l\varphi$ ($l\equiv\pm 1 \bmod p$) explicitely have been suppressed. Not before these exclusions should have been raised and the congruences (29) should have been completed with $l\equiv\pm 1 \bmod p$, (mechanically spoken, not until the buckling problem treated in sect. 3 should have been extended to such cases, in which the supplementary buckling loadsystem — eventually not in equilibrium itself — is balanced by a suitable radial loadsystem of the type ($a\cos\varphi+b\sin\varphi$)), complete conformity between eqs. (24) can exist. Obviously this conformity is formally indispensable if the theorems connected with eq. (32) shall be

transferred to eq. (24); especially if it is stated that any arbitrary function $V(\varphi)$ can be expanded into a series of the characteristic functions of (24). On the other hand it is seen at once that two characteristic functions U^{q_1} and U^{q_2} relating to two different congruences $l = q_1 \bmod p$ and $l = q_2 \bmod p$, never will have any cos. or sin. term in common. Consequently the statement can be made — which is essential for the method to be developed hereafter — that any arbitrary Fourier-series, containing only terms $\cos l\varphi$ and $\sin l\varphi$ for which $l \equiv q_1 \bmod p$, can be expanded in the characteristic functions $U_q{}^1$ relating to the same congruence $l = q_1 \bmod p$. For if we put

$$V^{q_1}(\varphi) = \Sigma a_k U_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where in the righthand side the summation provisionally must be extended over *all* characteristic functions of (32), the coefficient a_k is given by

$$a_k = N_0(\varphi) \int_0^{2\pi} V^{q_1}(\varphi)\, U_k(\varphi)\, d\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

provided that the characteristic functions have been normalized such that

$$\int_0^{2\pi} N_0(\varphi)\, U_k^2(\varphi)\, d\varphi = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Obviously the righthand side of (35) is zero for every function $U_k^{q_2}$, which does not belong to the class of characteristic functions, defined by the congruence $l \equiv q_1 \bmod p$. For (possibly apart from a constant term), the product $N_0(\varphi) U_k^{q_2}$ consists of the same terms as $U_k^{q_2}$ itself (comp. (24)) and therefore has no terms in common with $V^{q_1}(\varphi)$, which was supposed to consist of the same cos and sin-functions as $U_k^{q_1}$. The expansion (34) therefore reduces to

$$V^{q_1}(\varphi) = \Sigma a_k U_1^{q_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

5. *The iterative method.* It has already been stated that every congruence (29) defines a distinct infinite class of characteristic functions U_k and a set of corresponding characteristic numbers λ_k ($|\lambda_1| < |\lambda_2| < |\lambda_3, \ldots$). To find the smallest of these numbers we start with an arbitrary function $V_1(\varphi)$, containing only such cos and sin terms as appear in the functions U_k, so that it can be represented by

$$V_1 = \sum_1^{\infty} a_k U_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

From this function we derive another one, defined by the differential equation

$$V_2'' + V_2 = \frac{N_0 r^2}{EI} V_1 + C = \sum_1^{\infty} \frac{a_k r^2 N_0 U_k}{EI} + C \quad . \quad . \quad . \quad (39)$$

and the condition that it does not contain terms of the type $\alpha + \beta \cos \varphi + \gamma \sin \varphi$ (C being a constant to be determined a posteriori). Obviously V_2 is represented by

$$V_2 = \Sigma \frac{a_k U_k}{\lambda_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

for on account of the relations

$$U_k'' + U_k = \frac{\lambda_k N_0 r^2}{EI} U_k + l_k \quad (k = 1, 2, \ldots) \quad . \quad . \quad . \quad (41)$$

(comp. (24)) eq. (39) can be written as

$$V_2'' + V_2 = \left[\sum_1^\infty \frac{a_k U_k}{\gamma_k}\right]'' + \left[\sum_1^\infty \frac{a_k U_k}{\lambda_k}\right] + C - \sum_1^\infty \frac{a_k l_k}{\lambda_k} \quad . \quad . \quad (42)$$

By the appointment, that C must be chosen such that $C - \sum_1^\infty a_k l_k / \lambda_k$ is zero, (40) follows from (42). If now analogously a third function V_3 is derived from V_2, a.s.o. we find successively

$$V_3 = \sum_1^\infty \frac{a_k U_k}{\lambda_k{}^2}, \; V_3 = \sum_1^\infty \frac{a_k U_k}{\lambda_k{}^3}, \ldots V_n = \sum_1^\infty \frac{a_k U_k}{\lambda_k{}^{n-1}}, \; V_{n+1} = \sum_1^\infty \frac{a_k U_k}{\lambda_k{}^{n+1}}$$

and therefore

$$\lim_{n \to \infty} \frac{V_n}{V_{n+1}} = \lim_{n \to \infty} \frac{a_k U_k}{\lambda_k{}^{n-1}} : \sum_1^\infty \frac{a_k U_k}{\lambda_k{}^n} = \lim_{n \to \infty} \frac{U_1 + \sum_2^\infty \left(\frac{\lambda_1}{\lambda_k}\right)^{n-1} \frac{a_k}{a_1} U_k}{U_1 + \sum_2^\infty \left(\frac{\lambda_1}{\lambda_k}\right)^n \frac{a_k}{a_1} U_k} \lambda_1 = \lambda_1 \quad (43)$$

In practice the iteration process can be stopped as soon as two consecutive functions V_m and V_{m+1} are practically similar, λ_1 then being approximated by the slightly varying factor of similarity $V_m(\varphi) : V_{m+1}(\varphi)$. The approximation can be refined and the numerical work efficiently reduced by putting [3])

$$\text{either } \lambda_1 \backsim \frac{\int_0^{2\pi} N_0 V_m V_{m+1} \, d\varphi}{\int_0^{2\pi} N_0 V_{m+1}^2 \, d\varphi} \text{ or } \lambda_1 \backsim \frac{\int_0^{2\pi} N_0 V_m^2 \, d\varphi}{\int_0^{2\pi} N_0 V_m V_{m+1} \, d\varphi} \quad . \quad (44)$$

By using these formulae the iteration can be stopped at very low values of the index m ($m = 1$ or 2).

The iterative method is not restricted to the calculation of the smallest characteristic number λ, but can as well be used if λ_2 (or any higher characteristic) number should be required. We only have to start with an

[3]) Comp. footnote 4.

initial function V_1, which does not contain the first characteristic function U_1. The way, in which an arbitrary function V_1 can (approximately) be "cleaned" from U_1, and the way in which the iterations of such a nearly cleaned function V_1 can themselves be freed from rests of U_1, which they might contain, will not be treated here [4]).

6. *The iteration scheme.* The only thing still to be described is the scheme along which the iteration has to be performed in fact. Let, to fix our mind, N be given in the simple form

$$N = \lambda N_0 = \lambda (1 + \varepsilon_k \cos k\varphi) \quad \ldots \ldots \quad (45)$$

and let

$$V_1 = \cos l\varphi \quad \ldots \ldots \ldots \quad (46)$$

be our starting function, l representing any number, which is compatible with our problem. Then V_2 must satisfy the equation

$$\left. \begin{aligned} V_2'' + V_2 = -\frac{r^2}{EI} \cos l\varphi (1 + \varepsilon_k \cos k\varphi) + C = -\frac{r^2}{EI} \Big[\cos l\varphi + \\ + \frac{\varepsilon_k}{2} \cos (l+k)\varphi + \frac{\varepsilon_k}{2} \cos (l-k)\varphi \Big] + C \end{aligned} \right\} \quad (47)$$

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[4]) Comp. f. i. J. J. KOCH: Eenige toepassingen van de leer der eigenfuncties op vraagstukken uit de Toegepaste Mechanica, Doctor Thesis 1929 Delft, or C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, III, **14**, **15**, Springer 1939, Berlin. In these treatises only the first mentioned approximation is discussed but the second one can analogously be verified. It may be stated here without proof, that from the four approximations

$$\lambda_1 = \frac{\int_0^{2\pi} N_0 V_{m-1}^2 \, d\varphi}{\int_0^{2\pi} N_0 V_{m-1} V_m \, d\varphi}; \quad \lambda_1 = \frac{\int_0^{2\pi} N_0 V_{m-1} V_m \, d\varphi}{\int_0^{2\pi} N_0 V_m^2}; \quad \lambda_1 = \frac{\int_0^{2\pi} N_0 V_m^2 \, d\varphi}{\int_0^{2\pi} N_0 V_m V_{m+1} \, d\varphi};$$

$$\lambda_1 = \frac{\int_0^{2\pi} N_0 V_m V_{m+1} \, d\varphi}{\int_0^{2\pi} N_0 V_{m+1}^2 \, d\varphi}$$

each one excels the foregoing in accuracy provided that all characteristic numbers are positive. If negative characteristic numbers occur, the statement becomes doubtful with respect to the third and second approximation, though *in general* it may be expected that in this case too the statement holds true.

from which we deduce

$$\frac{r^2}{(l^2-1)\,EI}\left[\cos l\varphi + \frac{\varepsilon_k}{2}\,\frac{(l^2-1)}{(l+k)^2-1}\cos(l+k)\varphi + \frac{\varepsilon_k}{2}\,\frac{(l^2-1)}{(l-k)^2-1}\cos(l-k)\varphi + C\right] \quad . \; . \; . \quad (48)$$

The condition that no constant term shall be present in V_2 requires either $C=0$ (if $k \neq l$) or $C = -\frac{\varepsilon_k}{2}(l^2-1) : [(l-k)^2-1]$ (if $k=l$).

Furthermore it may be emphasized that the factor $r^2 : (l^2-1)EI$ represents the constant normal force N in the crosssections of a ring subjected to the critical normal pressure $q = (l^2-1)EI/r^3$. By placing this factor before the [ ] brackets a convenient comparison is made possible with this case of buckling. For if we write

$$V_3 = \left(\frac{r^2}{(l^2-1)\,EI}\right)^2 V_3^*(\varphi); \ldots . \; V_n = \left(\frac{r^2}{(l^2-1)\,EI}\right)^{n-1} V_n^*(\varphi) \quad (49)$$

the characteristic number λ_1 will be represented by

$$\lambda_1 = \lim_{n\to\infty}\frac{V_n}{V_{n+1}} = \frac{(l^2-1)\,EI}{r^2}\lim_{n\to\infty}\frac{V_n^*}{V_{n+1}^*} \quad . \; . \; . \; . \quad (50)$$

and the corresponding critical load by

$$q = \frac{\lambda_1}{r}(1+\varepsilon_k\cos k\varphi) = \frac{(l^2-1)\,EI}{r^3}(1+\varepsilon_k\cos k\varphi)\lim_{n\to\infty}\frac{V_n^*}{V_{n+1}^*} \quad . \quad (51)$$

If we compare the functions V_1^* ($\equiv V_1$) and V_2^* we see that V_2^* (apart from the constant C) is composed of the unchanged function V_1^* and of well defined multiples of the two functions $\cos(l+k)\varphi$ and $\cos(l-k)\varphi$. The function V_3^* consequently can be deduced from V_2^* by applying the same law of development to each of the individual terms of V_2^*; and obviously the same holds for any following iteration $V_2^* \to V_3^*$, $V_3^* \to V_4^*$ a.s.o.

Table I shows for $k=4$, $\varepsilon_k = \frac{1}{2}$, $l=2$ now the successive iterations best can be performed.

The columns of Table I are provided with the superscriptions $\cos 2\varphi$, $\cos 6\varphi$, ... $\cos(2+(n-1)4)\varphi$, and the corresponding factors $(2^2-1) : (2^2-1)$, $(2^2-1) : (6^2-1)$ a.s.o. As starting function has been chosen $\cos 2\varphi$; it is represented by the number 1 in the first row and first column. As stated before this term gives in the first iteration rise to certain multiples of $\cos -2\varphi$ and $\cos 6\varphi$ (comp. 48). If, provisionally, only attention is given to the multiplicatior $\varepsilon_k/2 = 0{,}25$, the number $0{,}25 \times 1$ should be shifted one place to the left and one place to the right. As $\cos -2\varphi = \cos 2\varphi$, the numbers destined for column $\cos -2\varphi$ (not present in table I), can be placed in the column $\cos 2\varphi$, and this indeed has been done as can be seen from the first number in the second row in table I. Thereupon the

numbers placed above the first short horizontal line in Table I have been summed up to find the coefficients of the product $W_1^* = N_0 V_1^*$ (comp. eq. (47)). Multiplication of these results with the multiplicators mentioned in the heads of the columns gives the coefficients of the required function V_2^*

$$V_2^* = 1{,}25000 \cos 2\varphi + 0{,}02143 \cos 4\varphi. \quad . \quad . \quad . \quad . \quad (52)$$

TABLE I.

$l=2$, $k=4$, $\varepsilon_k=\frac{1}{2}$	$\frac{2^2-1}{2^2-1}=1$ cos 2φ	$\frac{2^2-1}{6^2-1}=\frac{3}{35}$ cos 6φ	$\frac{2^2-1}{10^2-1}=\frac{1}{33}$ cos 10φ	$\frac{2^2-1}{14^2-1}=\frac{1}{65}$ sos 14φ	
$V_1 = V_1^*$	1.00000				
	0.25000	0.25000			
$W_1^* = N_0 V_1^*$	1.25000	0.25000			
V_2^*	1.25000	0.02143			
	0.31250	0.31250			
	0.00536		0.00536		
$W_2^* = N_0 V_2^*$	1.56786	0.33393	0.00536		
V_3^*	1.56786	0.02862	0.00017		
	0.39197	0.39197			
	0.00716	0.00004	0.00716	0.00004	
$W_3^* = N_0 V_3^*$	1.96699	0.42063	0.00733	0.00004	
V_4^*	1.96699	0.03605	0.00022	0.00000	
	0.49175	0.49175			
	0.00901	0.00006	0.00901	0.00006	
$W_4^* = N_0 V_4^*$	2.46775	0.52786	0.00923	0.00006	
V_5^*	2.46775	0.04525	0.00028	0.00000	
V_3^* / V_4^*	0.7971	0.7939	0.7727	—	
V_4^* / V_5^*	0.7910	0.7967	0.7856	—	
$W_1^* V_1^*$	1.25000	0.00000	0.00000	0.00000	$\frac{\Sigma_1}{\Sigma_2} = 0.79727$
$W_1^* V_2^*$	1.56250	0.00536	0.00000	0.00000	
$W_2^* V_2^*$	1.95983	0.00710	0.00000	0.00000	$\frac{\Sigma_1}{\Sigma_2} = 0.79708$
$W_2^* V_3^*$	2.45818	0.00956	0.00000	0.00000	
$W_3^* V_3^*$	3.08396	0.01204	0o00000	0.00000	$\frac{\Sigma_1}{\Sigma_2} = 0.79708$
$W_3^* V_4^*$	3.86905	0.01516	0.00000	0.00000	
$W_4^* V_4^*$	4.85404	0.01903	0.00000	0.00000	$\frac{\Sigma_1}{\Sigma_2} = 0.79708$
$W_4^* V_5^*$	6.08979	0.02389	0.00000	0.00000	

(In controling this step, the reader has to pay due attention to the fact, that the factor $(l^2-1) : [(l-k)^2-1]$ of the third term between brackets [ ] of eq. (48) has the value one in virtue of $k = -2l$). The coefficients of V_2^* again are shifted one place to the left and one place to the right after

having been multiplied with the factor $\varepsilon_k/2 = 0{,}25$, under the understanding that every number destined for column cos -2φ be placed in the column cos 2φ. The numbers thus obtained (and placed beneath the first long horizontal line of Table I) are added, — giving as result the coefficients of $W_2^* = N_0 V_2^*$ — and these coefficients in their turn have been multiplied by the factors 1, 3/35, 1/33 .... In this way we find

$$V_2^* = 1{,}56786 \cos 2\varphi + 0{,}02862 \cos 6\varphi + 0{,}00017 \cos 10\varphi . \quad . \quad . \quad (53)$$

As will be seen from Table I the iteration has been pursued up to V_5^*. Then the quotients of the corresponding coefficients of V_3^* and V_4^* (resp. of V_4^* and V_5^*) have been calculated and it may be stated that a high degree of similarity between these coefficients exists. Clearly the factor of proportionality is best approximated by the figures in the first column and therefore we put:

$$\lambda_1 = 0{,}7971 \cdot \frac{3\,EI}{r^2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

If we should have used the second of the formulae (44)

$$\left.\begin{aligned} \lambda_1 = \frac{\int_0^{2\pi} N_0 \cdot V_m^2 \, d\varphi}{\int_0^{2\pi} N_0 \, V_m \, V_{m+1} \, d\varphi} &= \frac{(l^2-1)\,EI}{r^2} \, \frac{\int_0^{2\pi} N_0 \, V_m^{2*} \, d\varphi}{\int_0^{2\pi} N_0 \, V_m^* \, V_{m+1}^* \, d\varphi} = \\ &= \frac{3\,EI}{r^2} \, \frac{\int_0^{2\pi} W_m^* \, V_m^* \, d\varphi}{\int_0^{2\pi} W_m^* \, V_{m+1}^* \, d\varphi} \end{aligned}\right\} \quad (55)$$

we should have found (with $m = 1, 2, 3,$ and 4): $\lambda_1 = 0{,}7973$; $0{,}7971$; $0{,}7971$; $0{,}7971 \dfrac{3EI}{r^2}$ respectively [5]). It appears that, even if we had stopped the iteration with V_2^*, the relative error in λ_1 would have been less than 1/4000.

Some additional remarks with respect to the iterative process have to be made. Firstly it must be stated, that in the case just treated the iteration could as well have been started with $V_1 \equiv V_1^* \equiv \sin l\varphi$, l again represen-

[5]) The work to be done in computing these successive approximations for λ_1 is represented in the last 8 rows of Table I. Obviously an integral such as $\int_0^{2\pi} W_1^* \, V_2^* \, d\varphi$ can be evaluated by multiplying the corresponding coefficients of W_1^* and V_2^* and by summing up the so-acquired products.

ting a number compatible with our problem. Then we should have found [6])

$$V_2 = \frac{r^2}{(l^2-1)\,EI}\left[\sin l\varphi + \frac{\varepsilon_k}{2}\,\frac{l^2-1}{(l+k)^2-1}\sin(l+k)\varphi + \frac{\varepsilon_k}{2}\,\frac{l^2-1}{(l-k)^2-1}\sin(l-k)\varphi\right] \qquad (56)$$

The same scheme of calculation as given in Table I can be used here, provided that the superscriptions cos 2φ, cos 6φ ... be replaced by sin 2φ, sin 6φ ...; it must, however, be put in mind, that the transition of a term $\sin(l-k)\varphi$ from the column $\sin(l-k)\varphi$ to the column $\sin|l-k|\varphi$ involves the introduction of a factor (-1). It would be found that the characteristic value λ_1, calculated in this way is greater than the first one. If $V_1 = \cos l\varphi + \sin l\varphi$ had been chosen as the starting function one is invariably led to the first characteristic value (54) and to the corresponding characteristic function. A slight modification would occur if the special case $N = \lambda\,(1 + \varepsilon_2 \cos 2\varphi)$, $(k = 2)$ would be examined. It may be left to the reader to iterate once with $V_1^* = \cos 2\varphi$ and once with $V_1^* = \sin 2\varphi$ as starting functions. He will be led to the same characteristic number λ_1 and to two linearly independent characteristic functions!

A second remark refers to the fact, that a slight complication appears if $N = \lambda\,(1 + \varepsilon_k \cos k\varphi)$ is replaced by $N = \lambda\,(1 + \varepsilon_k \sin k\varphi)$. Obviously we then have to start with a function composed of cos and sin terms, for instance $V_1 = \alpha \cos l\varphi + \beta \sin l\varphi$. The first iteration, defined by

$$V_2'' + V_2 = -\frac{r^2}{EI}\left[\alpha \cos l\varphi + \beta \sin l\varphi\right]\left[1 + \varepsilon_k \sin k\varphi\right] + C \quad . \quad (57)$$

then proves to be

$$V = \frac{\alpha r^2}{(l^2-1)\,EI}\left[\cos l\varphi + \frac{\varepsilon_k}{2}\,\frac{l^2-1}{(l+k)^2-1}\sin(l+k)\varphi - \frac{\varepsilon_k}{2}\,\frac{l^2-1}{(l-k)^2-1}\sin(l-k)\varphi\right] + \frac{\beta r^2}{(l^2-1)EI}\left[\sin l\varphi - \frac{\varepsilon_k}{2}\,\frac{l^2-1}{(l+k)^2-1}\cos(l+k)\varphi + \frac{\varepsilon_k}{2}\,\frac{l^2-1}{(l-k)^2-1}\cos(l-k)\varphi\right] + C. \qquad (58)$$

Again the constant C is determined by the condition, that ultimately no constant term in V_2 is allowed.

Table II represents the scheme of iteration, adapted to this case, with $\varepsilon_k = 1$, $k = 2$, and $\cos 2\varphi + \sin 2\varphi$ as starting function. It has, for the first three iterations been indicated by asterisks and cross-signs how the

[6]) The constant of integration C of eq. (48) vanishes here under all circumstances as the sine-functions in (56) for no single value of k give rise to a constant term.

TABLE II.

$l=2$ $k=2$ $\varepsilon_k=1$	$\frac{2^2-1}{2^2-1}=1$		$\frac{2^2-1}{4^2-1}=\frac{3}{35}$		$\frac{2^2-1}{6^2-1}=\frac{3}{35}$		$\frac{2^2-1}{8^2-1}=\frac{1}{33}$		$\frac{2^2-1}{10^2-1}=\frac{1}{33}$	
	$\cos 2\varphi$	$\sin 2\varphi$	$\cos 4\varphi$	$\sin 4\varphi$	$\cos 6\varphi$	$\sin 6\varphi$	$\cos 8\varphi$	$\sin 8\varphi$	$\cos 10\varphi$	$\sin 10\varphi$
V_1^*	1.00000*	1.00000x	—0.50000x	0.50000*						
W_1^*	1.00000	1.00000	—0.50000	0.50000xx						
V_2^*	1.00000* 0.05000xx	1.00000x 0.05000**	—0.10000** —0.50000x	1.00000xx 0.50000*	—0.05000xx	— 0.05000**				
W_2^*	1.05000	1.05000	—0.60000	+0.60000	—0.05000	—0.05000				
V_3^*	1.05000* 0.06000xx	1.05000x 0.06000xx	—0.12000** —0.52500x —0.00214xxx	0.12000xx 0.52500 0.00214***	—0.00429*** —0.06000xx	—0.00429xxx —0.06000**	0.00214xxx	—0.00214***		
W_3^*	1.11000	1.11000	—0.64714	0.64714	—0 06429	—0.06429	0.00214	—0.00214		
V_4^*	1.11000 0.06471	1.11000 0.06471	—0.12943 —0.55500 —0.00276	0012943 0.55500 0.00276	—0.00551 —0.06471 —0.00005	—0.00551 —0.06471 —0.00005	0.00010 0.00276	—0.00010 —0.00276	0.00005	0.00005
W_4^*	1.17471	1.17471	—0.68719	0.63719	—0.07027	—0.07027	0.00286	—0.00286	0.00005	0.00005
V_5^*	1.17471	1.17471	—0.13744	0.13744	—0.00600	—0.00600	0.00014	—0.00014		
V_4^*/V_5^*	0.946	0.946	0.942	0.942	0.918	0.918	—	—	—	
$W_1^*\ V_1^*$ $W_1^*\ V_2^*$	1.00000 1.00000	1.00000 1 00000	0.05000	0.05000			$\Sigma_1 \equiv 2.00000$ $\Sigma_2 = 2.10000$		$\frac{\Sigma_1}{\Sigma_2}=0.95238$	
$W_2^*\ V_2^*$ $W_2^*\ V_3^*$	1.05000 1.10250	1.05000 1.10250	0.06000 0.07200	0.06000 0.07200	0.00000 0.00021	0.00000 0.00021	$\Sigma_1 = 2.22000$ $\Sigma_2 = 2.34942$		$\frac{\Sigma_1}{\Sigma_2}=0.94491$	
$W_3^*\ V_3^*$ $W_3^*\ V_4^*$	1.16550 1.23210	1.16550 1.23210	0.07766 0.08373	0.07766 0.08373	0.00028 0.00035	0.00028 0.00035	$\Sigma_1 = 2.48688$ $\Sigma_2 = 2.63236$		$\frac{\Sigma_1}{\Sigma_2}=0.94473$	
$W_4^*\ V_4^*$ $W_4^*\ V_3^*$	1.30393 1.37994	1.30393 1.37994	0.08894 0.09445	0.08894 0.09445	0.00039 0.00042	0.00039 0.00042	$\Sigma_1 = 2.78552$ $\Sigma_2 = 2.94962$		$\frac{\Sigma_1}{\Sigma_2}=0.94470$	

different figures have to be shifted. As to the iteration it will be seen, that the first figure of V_2^* has been placed (multiplied by $\varepsilon_2/2 = 1/2$) on the fourth place of the next row; the second figure multiplied by $-\varepsilon_2/2$ on the third place; the third one, multiplied by $\varepsilon_2/2$ and $-\varepsilon_2/2$ respectively on the sixth and second places; the fourth one, multiplied by $-\varepsilon_2/2$ and $\varepsilon_2/2$ respectively on the fifth and first place. Then the corresponding figures of the two rows under consideration have been added, and the results have been multiplied by the factors inserted in the headings of the columns. The iteration has been stopped with V_5^* and it is seen by comparison of V_4^* and V_5^*, that the required smallest characteristic number λ_1 (with considerable approximation) can be represented by $\lambda = 3EI/r^2 \,.\, 0{,}946$. The second formula (44) leads for $m = 1, 2, 3, 4$ to the following values

$$\lambda_1 = 0{,}95238, \qquad 0{,}94491, \qquad 0{,}94473, \qquad 0{,}94470$$

and it is confirmed again that much labour can be saved by the use of this formula.

7. *The compressive force* $N = \lambda N_0 = \lambda(1 + 2\cos k\varphi)$. In this section the numerical results are collected, with regard to the special normal force distribution $N = \lambda\,(1 + 2\cos k\varphi)$, up to $k = 12$. The example has chiefly been chosen to get an insight in the influence of a strong fluctuation of N with respect to its mean value on the buckling force of the ring. As has already been stated in sects. 3 and 4 that for every value of k the buckling problem is split up into $(k-1)/2$ or $k/2$ separate problems connected with the congruences $l \equiv \pm q \bmod k$ $\left(q = 0, 2, 3 \ldots \frac{k-1}{2} \text{ or } \frac{k}{2}\right)$; the characteristic

TABLE III. Giving the values of $\dfrac{\lambda r^2}{(l^2-1)\,EI}$ if $N = \lambda\,(1 + 2\cos k\varphi)$.

$k \backslash l$	2	3	4	5	6	7	8	9	10	11	12
2	0.832	—	—	—	—	—	—	—	—	—	—
3	—	0.812	—	—	—	—	—	—	—	—	—
4	0.489	—	0.808	—	—	—	—	—	—	—	—
5	0.708	—	—	0.802	—	—	—	—	—	—	—
6	0.810	0.488	—	—	0.801	—	—	—	—	—	—
7	0.866	0.634	—	—	—	0.799	—	—	—	—	—
8	0.899	0.728	0.487	—	—	—	0.799	—	—	—	—
9	0.922	0.789	0.598	—	—	—	—	0.798	—	—	—
10	0.937	0.831	0.679	0.486	—	—	—	—	0.798	—	—
11	0.948	0.862	0.738	0.577	—	—	—	—	—	0.797	—
12	0.957	0.885	0.782	0.647	0.486	—	—	—	—	—	0.797

functions of each of these problems is built up of terms cos $l\varphi$ and sin $l\varphi$ in which l is determined by one of these congruences.

Table III contains all values of $\lambda^2 : (l^2-1)EI$ for $2 \leqq \frac{k}{l} \leqq 12$. It may be emphasized that for any prescribed number l the smallest value of λ corresponds to $k = 2l$. This fact becomes explicable by the consideration that for $k = 2l$ the angles *both* of maximum and minimum deflection coincide with those of maximum compressive force N.

8. *The compressive force* $N = \lambda N_0 = \lambda\ (1 + \varepsilon_2 \cos 2\varphi + \varepsilon_4 \cos 4\varphi)$, $\varepsilon_2 = 2$, $\varepsilon_4 = 1$; *the second iteration.* As an illustration how to handle the iterative method if the compressive normal force has a more complicated form like $N = \lambda(1 + 2 \cos 2\varphi + 4\varphi)$ and how to act if apart from the first characteristic function the second one should be required, we insert in this section Tables IV*a*. and IV*b*. from which all necessary data can be borrowed. The starting function V_1^* is represented by cos 2φ. This term has to be shifted one place to the right and one place to the left after multiplication with the factor $\varepsilon_2/2 = 1$ on account of the term ε_2 cos 2φ in N. The shifting to the left gives a term in the column cos 0φ which must be suppressed (comp. sect. 6). Furthermore the starting term cos 2φ has to be shifted two places to the right and to the left after multiplication with the factor $\varepsilon_4/2 = 1/2$ on account of the term ε_2 cos 4φ in N. The shifting to the left provides us with a term in the column cos -2φ, which therefore must be placed in the column cos 2φ. Summation of all terms occurring in the different columns gives the coefficients of the function W_1^*. Finally these coefficients have to be multiplied by the factors 1, 1/5, 3/35 ... mentioned in the heads of the columns, to obtain the coefficients of the first iteration V_2^*. Analogously the terms of V_2^* have to be shifted *one* place to the right and left after multiplication with the factor $\varepsilon_2/2$ and *two* places to the right and left after multiplication with the factor $\varepsilon_4/2$, with due regard to the fact, that every number in the column cos 0. φ has to be suppressed and every number in the column cos -2φ has to be transported to the column cos 2φ. Summing up all terms in the different columns provides us with the coefficients of W_2^*, whereas multiplication of these coefficients with their respective multiplicators 1, 1/15 ... furnishes the coefficients of V_3^*; a.s.o. The iterative process has been carried on to the sixth iteration V_7^*, though the computation of the first characteristic number λ_1 in itself would require no more than the iterations V_2^* and V_3^*. By proceeding as far as V_3^*, however, great accuracy is obtained in the first characteristic function U_1 which is approximated by

$$U_1 \backsim V_7^* = \left.\begin{array}{l} 18{,}493 \cos 2\varphi + 2{,}627 \cos 4\varphi + 0{,}650 \cos 6\varphi + \\ \qquad + 0{,}058 \cos 8\varphi + 0{,}007 \cos 10\varphi \end{array}\right\} \quad (59)$$

TABLE IVa.

$N = \lambda\ [1 + 2\cos 2\varphi + \cos 4\varphi]$; $\varepsilon_2 = 2$, $\varepsilon_4 = 1$.

	1 cos 2 φ	1/5 cos 4 φ	3/35 cos 6 φ	1/21 cos 8 φ	1/33 cos 10 φ	3/143 cos 12 φ
V_1^*	1.000*o*					
		1.000*or*				
			0.500*oor*			
	5.000*ool*					
W_1^*	1.500	1.000	0.500			
V_2^*	1.500*o*	0.200*o*	0.043*x*			
		1.500*or*	0.200*or*	0.043*xr*		
	0.200*ol*	0.043*xl*				
			0.750*oor*	0.100*oor*	0.022*xxr*	
	0.022*xxl*					
	0.750*ool*					
W_2^*	2.472	1.743	0.993	0.143	0.043	
V_3^*	2.472	0.349	0.085	0.007	0.001	
		2.472	0.349	0.085	0.007	0.001
	0.349	0.085	0.007	0.001		
			1.236	0.175	0.043	0.004
	0.043	0.004	0.001			
	1.236					
W_3^*	4.100	2.910	1.678	0.268	0.051	0.005
V_4^*	4.100	0.582	0.144	0.012	0.002	0.000
		4.100	6.582	0.144	0.012	0.002
	0.582	0.144	0.012	0.002		
			2.052	0.291	0.072	0.006
	0.072	0.006	0.001			
	2.050					
W_4^*	6.704	4.832	2.789	0.449	0.086	0.008
V_5^*	6.704	0.966	0.239	0.021	0.003	0.000
		6.704	0.966	0.239	0.021	0.003
	0.966	0.239	0.021	0.003		
		3.352	0.483	0.120	0.011	
	0.120	0.011	0.002			
	3.352					
W_5^*	11.142	7.920	4.580	0.746	0.144	0.014
V_6^*	11.142	1.584	0.392	0.036	0.004	0.000
		11.142	1.584	0.392	0.036	0.004
	1.584	0.392	0.036	0.004		
			5.571	0.792	0.196	0.018
	0.196	0.018	0.002			
	5.571					
W_6^*	18.493	13.136	7.585	1.224	0.236	0.022
V_7^*	18.493	2.627	0.650	0.058	0.007	0.000
		18.493	2.627	0.650	0.058	0.007
	2.627	0.650	0.058	0.007		
			9.247	1.314	0.325	0.029
	0.325	0.029	0.004			
	9.247					
W_7^*	30.692	21.799	12.586	2.029	0.390	0.036
$W_7^* V_6^*$	341.96	34.53	4.93	6.07	$\Sigma_1 = 381.50$	$\frac{\Sigma_1}{\Sigma_2} = \lambda_1 = 0.60253$
$W_7^* V_7^*$	467.59	57.27	8.18	0.12	$\Sigma_2 = 633.16$	

TABLE IV*b*. $N = \lambda\ [1 + 2\cos 2\varphi + \cos 4\varphi]$. $\varepsilon_2 = 2$, $\varepsilon_4 = 1$.

	1 $\cos 2\varphi$	1/5 $\cos 4\varphi$	3/35 $\cos 6\varphi$	1/21 $\cos 8\varphi$	1/33 $\cos 10\varphi$	3/143 $\cos 12\varphi$
V_8^*	100.000					
$V_8^*\ W_7^*$	3069.200					
$-4.8474\ V_7^*$	−89.643	−12.724	−3.151	−0.281	−0.034	—
V_8^*	10.357	−12.734	−3.151	−0.281	−0.034	
		10.357	−12.734	−3.151	−0.281	−0.034
	−12.734	−3.151	−0.281	−0.034		
			5.179	−6.367	−1.576	−0.141
	−1.576	−0.141	−0.017			
	5.179					
W_8^*	1.226	−5.669	−11.004	−9.833	−1.891	−0.175
V_9^*	1.226	−1.134	−0.943	−0.468	−0.057	−0.004
$V_9^*\ V_7^*$	37.628	−24.720	−11.869	−0.950	−0.022	
$-0.000106\ V_7^*$	−0.002	—	—	—	—	
V_9^*	1.224	−1.134	−0.943	−0.468	−0.057	−0.004
		1.224	−1.134	−0.943	−0.468	−0.057
	−1.134	−0.943	−0.468	−0.057	−0.004	
			0.612	−0.567	−0.472	−0.234
	−0.472	−0.234	−0.029	−0.002		
	+0.612					
W_9^*	0.230	−1.087	−1.962	−2.037	−1.001	−0.295
V_{10}^*	0.2300	−0.2174	−0.1682	−0.0970	−0.0303	−0.0062
$V_{10}^* W_7^*$	7.059	−4.739	−2.117	−0.197	−0.012	—
$+0.0000095\ V_7^*$	0.0002					
V_{10}^*	0.2302	−0.2174	−0.1682	−0.0970	−0.0303	−0.0062
$\overline{V}_{11}^*$	0.04379	−0.04078	−0.03280	−0.01940	−0.00659	−0.00178
$\overline{V}_{12}^*$	0.008495	−0.007900	−0.006375	−0.003813	−0.001338	−0.000379
$\overline{V}_{11}^* : \overline{V}_{12}^*$	5.155	5.162	5.145	5.064	4.889	4.670

The characteristic number λ_1 has been computed with the aid of the first formula (44)

$$\left.\begin{aligned} \lambda_1 &= \frac{\int\limits_0^{2\pi} N_0\, V_6\, V_7\, d\varphi}{\int\limits_0^{2\pi} N_0\, V_7^2\, d\varphi} = \frac{3\,EI}{r^2}\,\frac{\int\limits_0^{2\pi} N_0\, V_6^*\, V_7^*\, d\varphi}{\int\limits_0^{2\pi} N_0\, V_7^{*2}\, d\varphi} = \\ &= \frac{3\,EI}{r^2}\,\frac{\int\limits_0^{2\pi} W_7^*\, V_6^*\, d\varphi}{\int\limits_0^{2\pi} W_7^*\, V_7^*\, d\varphi} = 0{,}60253\ \frac{3\,EI}{r^2} \end{aligned}\right\} \quad . \quad . \quad (60)$$

Table IV*b* is devoted to the calculation of λ_2. The starting function is represented by $V_8^* = 100 \cos 2\varphi$. The coefficient a_1 in the development of V_8^* into a series of the characteristic functions U

$$V_8^* = a_1 U_1 + a_2 U_2 + \dots \quad \dots \dots \dots (61)$$

is given by

$$\left. \begin{aligned} a_1 &= \frac{\int_0^{2\pi} N_0 V_8^* U_1 \, d\varphi}{\int_0^{2\pi} N_0 U_1^2 \, d\varphi} \text{ and approximately by} \\ a_1^* &= \frac{\int_0^{2\pi} N_0 V_8^* V_7^* \, d\varphi}{\int_0^{2\pi} N_0 V_7^{*2} \, d\varphi} = 4{,}847 \end{aligned} \right\} \quad . \; . \; (62)$$

If a_1 and U_1 would have been obtained *exactly*, the iteration process applied to $V_8^* - a_1 U_1$ obviously would lead to the enumeration of the *second* characteristic number λ_2. In reality both U_1 and a_1 are only approximately known and therefore, if the process is applied to the function $\overline{V}_8^* = V_8^* - a_1 V_7^*$ which to a certain (but small) amount contains the first characteristic function U_1 — serious difficulties are to be expected. For indeed, however small the contribution of U_1 in any starting function may be, the iteration process always and invariably leads to λ_1 if U_1 initially is present. These difficulties are surmounted by iterating V_9^* from $\overline{V}_8^*$, by cleaning V_9^* from $U_1 \infty V_7^*$ in the same way as V_8^* has been cleaned from U_1, and by repeating this combined iteration and cleaning process till two consecutive "cleaned" functions $\overline{V}_m^*$ and $\overline{V}_{m+1}^*$ are sufficiently proportional. Here too considerable abbreviation of the cipherwork can be attained by the use of the formulae (44) if the characteristic number λ_2 alone should be required, as will be seen from the results

$$\lambda_2 = 5{,}2602, \quad 5{,}1331, \quad 5{,}1234, \quad 5{,}1228 \quad . \; . \; . \; . \; (63)$$

which have been found by the first of the formulae (44) with $m = 8, 9, 10$ and 11 respectively. The second of these values already approximates λ_2 quite satisfactory.

The determination of the second characteristic function, however, requires the continuation up to V_{12}^*

$$\left. \begin{aligned} V_{12}^* = 0{,}008495 \cos 2\varphi - 0{,}007900 \cos 4\varphi - 0{,}006375 \cos 6\varphi - \\ - 0{,}003813 \cos 8\varphi - 0{,}001338 \cos 10\varphi - 0{,}000379 \cos 12\varphi. \end{aligned} \right\} (64)$$

9. *Negative characteristic numbers.* It has already been stated that dealing with buckling problems one has to expect negative characteristic values in all such cases, where the normal force N changes its sign. A negative value of λ interchanges the compressed and the stretched parts of the construction, and a sufficiently great negative λ therefore causes the buckling of the initially stretched parts. As an example it may be stated that with a compressive force $N = \lambda\,[1 + 4\cos 2\varphi]$ the *second* characteristic number proves to be negative, $\lambda_2 = -2{,}1133\,\frac{3EI}{r^2}$, whereas the first characteristic numbers λ_1 equals to $\lambda_1 = 0{,}6275\,.\,\frac{3EI}{r^2}$.

Geodesy. — *Equations for elastic solids in spherical coordinates, including the case that the temperature is not constant in space and that a gravitational force is working in the sense of the radius. Solution of these equations, also applicable to viscous fluids, for the general problem that the radial components P_ϱ and V_ϱ of the mass-forces and of the elastic displacements resp. the velocities, as well as the normal components σ_ϱ of the stresses on the spheres and the temperature θ are functions of ϱ multiplied by the same spherical harmonic, while the components P and V on the spheres of the mass-forces and of the elastic displacements resp. the velocities, and the shearing-stresses τ on the spheres are functions of ϱ multiplied by gradients of this spherical harmonic.* By F. A. VENING MEINESZ.

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§ 1. *Equations for elastic solids resp. for viscous fluids in orthogonal coordinates x, y, z; the temperature θ is assumed to be variable, but gravity to be constant over the area investigated.*

Before dealing with the matter in spherical coordinates we shall take it up in orthogonal coordinates. The Z axis is chosen contrary to the sense of gravity.

The conditions for the equilibrium of an element dx, dy, dz of the elastic solid and of the viscous fluid are in both cases

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_z}{\partial y}+\frac{\partial \tau_y}{\partial z}+P_x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_x}{\partial z}+\frac{\partial \tau_z}{\partial x}+P_y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

$$\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_y}{\partial x}+\frac{\partial \tau_x}{\partial y}+P_z+\alpha \varrho' g\, \theta=0 \quad . \quad . \quad . \quad . \quad (1c)$$

where $\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z$ are the normal and tangential stresses, P_x, P_y, P_z the components of the mass-forces, ϱ' the density, g the acceleration of gravity and α the temperature coefficient per unit of volume.

For the relation of the three components u, v, w of the elastic displacements to the stresses on the coordinate-planes and to the temperature θ we find for the elastic solid

$$\frac{\partial u}{\partial x}=\frac{1}{E}\left(\sigma_x-\frac{\sigma_y+\sigma_z}{m}\right)+\frac{1}{3}\alpha\theta=\frac{1}{2G}\left(\sigma_x-\frac{\sigma}{m+1}\right)+\frac{1}{3}\alpha\theta \quad (2a)$$

$$\frac{\partial v}{\partial y}=\frac{1}{E}\left(\sigma_y-\frac{\sigma_z+\sigma_x}{m}\right)+\frac{1}{3}\alpha\theta=\frac{1}{2G}\left(\sigma_y-\frac{\sigma}{m+1}\right)+\frac{1}{3}\alpha\theta \quad (2b)$$

$$\frac{\partial w}{\partial z}=\frac{1}{E}\left(\sigma_z-\frac{\sigma_z+\sigma_y}{m}\right)+\frac{1}{3}\alpha\,\theta=\frac{1}{2\,G}\left(\sigma_z-\frac{\sigma}{m+1}\right)+\frac{1}{3}\alpha\,\theta \qquad (2c)$$

where m is the coefficient of POISSON, and E and G the moduli of elasticity and shear which are related by the formula

$$G=\frac{m\,E}{2\,(m+1)}$$

Adding 2*a*, 2*b* and 2*c* together we find for the elastic divergence e

$$e=\frac{m-2}{m}\frac{\sigma}{E}+\alpha\,\theta=\frac{m-2}{m+1}\frac{\sigma}{2\,G}+\alpha\,\theta \quad . \; . \; . \; . \quad (2d)$$

in which

$$\sigma=\sigma_x+\sigma_y+\sigma_z$$

By means of these equations it is easy to express the stresses in the displacements and we find

$$\frac{\sigma_t}{2\,G}=\frac{\partial u}{\partial x}+\frac{e}{m-2}-\frac{(m+1)}{3\,(m-2)}\alpha\,\theta \quad . \; . \; . \; . \quad (3a)$$

$$\frac{\sigma_y}{2\,G}=\frac{\partial v}{\partial y}+\frac{e}{m-2}-\frac{(m+1)}{3\,(m-2)}\alpha\,\theta \quad . \; . \; . \; . \quad (3b)$$

$$\frac{\sigma_r}{2\,G}=\frac{\partial w}{\partial z}+\frac{e}{m-2}-\frac{(m+1)}{3\,(m-2)}\alpha\,\theta \quad . \; . \; . \; . \quad (3c)$$

$$\frac{\tau_x}{G}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3d)$$

$$\frac{\tau_y}{G}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3e)$$

$$\frac{\tau_z}{G}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3f)$$

Introducing these last formulae in the conditions for equilibrium 1*a*, 1*b* and 1*c* we obtain the equations for the elastic deformation brought about by these equilibrium conditions

$$\nabla^2 u+\frac{m}{m-2}\frac{\partial e}{\partial x}+\frac{P_x}{G}-\frac{2\,(m+1)}{3\,(m-2)}\alpha\frac{\partial\theta}{\partial x}=0 \quad . \; . \; . \; . \quad (4a)$$

$$\nabla^2 v+\frac{m}{m-2}\frac{\partial e}{\partial y}+\frac{P_y}{G}-\frac{2\,(m+1)}{3\,(m-2)}\alpha\frac{\partial\theta}{\partial y}=0 \quad . \; . \; . \; . \quad (4b)$$

$$\nabla^2 w+\frac{m}{m-2}\frac{\partial e}{\partial z}+\frac{P_z}{G}-\frac{2\,(m+1)}{3\,(m-2)}\alpha\frac{\partial\theta}{\partial z}+\frac{\alpha\,\varrho' g}{G}\theta=0 \quad . \quad (4c)$$

By differentiating these equations with regard to x, resp. y and z and by adding together we get

$$\nabla^2 e + \frac{(m-2)}{2(m-1)} \frac{1}{G} \left(\nabla P + \alpha \varrho' g \frac{\partial \theta}{\partial z} \right) - \frac{(m+1)}{3(m-1)} \alpha \nabla^2 \theta = 0 \quad (5)$$

or, by means of 2d

$$\nabla^2 \sigma + \frac{m+1}{m-1} \left(\nabla P + \alpha \varrho' g \frac{\partial \theta}{\partial z} \right) + \frac{4(m+1)}{3(m-1)} G \alpha \nabla^2 \theta = 0 \quad . \quad (6)$$

In these formulae P is the mass-force.

Finally we find by multiplying the equations 4 by 2G, by differentiating them with regard to x resp. y and z and by combining them with 5 and with the equations 3 to which the operation $\Delta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ has first been applied

$$\left. \begin{aligned} \nabla^2 \sigma_x + \frac{m}{m+1} \frac{\partial^2 \sigma}{\partial x^2} + 2 \frac{\partial P_x}{\partial x} + \frac{1}{m-1} \left(\nabla P - \alpha \varrho' g \frac{\partial \theta}{\partial z} \right) + \\ + \frac{2(m+1)}{3(m-1)} G \alpha \nabla^2 \theta + \frac{2}{3} G \alpha \frac{\partial^2 \theta}{\partial x^2} = 0, \end{aligned} \right\} \quad (7a)$$

$$\left. \begin{aligned} \nabla^2 \sigma_y + \frac{m}{m+1} \frac{\partial^2 \sigma}{\partial y^2} + 2 \frac{\partial P_y}{\partial y} + \frac{1}{m-1} \left(\nabla P - \alpha \varrho' g \frac{\partial \theta}{\partial z} \right) + \\ + \frac{2(m+1)}{3(m-1)} G \alpha \nabla^2 \theta + \frac{2}{3} G \alpha \frac{\partial^2 \theta}{\partial y^2} = 0, \end{aligned} \right\} \quad (7b)$$

$$\left. \begin{aligned} \nabla^2 \sigma_z + \frac{m}{m+1} \frac{\partial^2 \sigma}{\partial z^2} + 2 \frac{\partial P_z}{\partial z} + \frac{1}{m-1} \left(\nabla P - \alpha \varrho' g \frac{\partial \theta}{\partial z} \right) + \\ + \frac{2(m+1)}{3(m-1)} G \alpha \nabla^2 \theta + \frac{2}{3} G \alpha \frac{\partial^2 \theta}{\partial z^2} + 2 \alpha \varrho' g \frac{\partial \theta}{\partial z} = 0, \end{aligned} \right\} \quad (7c)$$

$$\nabla^2 \tau_x + \frac{m}{m+1} \frac{\partial^2 \sigma}{\partial y \partial z} + \frac{\partial P_z}{\partial y} + \frac{\partial P_y}{\partial z} + \frac{2}{3} G \alpha \frac{\partial^2 \theta}{\partial y \partial z} + \alpha \varrho' g \frac{\partial \theta}{\partial y} = 0, \quad (7d)$$

$$\nabla^2 \tau_y + \frac{m}{m+1} \frac{\partial^2 \sigma}{\partial x \partial z} + \frac{\partial P_x}{\partial z} + \frac{\partial P_z}{\partial x} + \frac{2}{3} G \alpha \frac{\partial^2 \theta}{\partial x \partial z} + \alpha \varrho' g \frac{\partial \theta}{\partial x} = 0, \quad (7e)$$

$$\nabla^2 \tau_z + \frac{m}{m+1} \frac{\partial^2 \sigma}{\partial x \partial y} + \frac{\partial P_y}{\partial x} + \frac{\partial P_x}{\partial y} + \frac{2}{3} G \alpha \frac{\partial^2 \theta}{\partial x \partial y} = 0 \quad . \quad . \quad . \quad . \quad (7f)$$

These equations are the comptability conditions which, together with the equilibrium conditions 1 and the boundary conditions control the stresses in an elastic solid.

In case we neglect the elastic deformations and — in accord with BOUSSINESQ's results [1]) — the variations of density, except in so far as

[1]) Théorie analytique de la chaleur, II, 173 (1902).

they modify the action of gravity, we find the same equations for a viscous fluid, but u, v and w now represent the components of the velocity, θ has to be replaced by $\frac{\partial \theta}{\partial t}$ except in the gravity term of equation 1c, the components P_x, P_y, P_z include the d'Alembert terms $\varrho' \frac{du}{dt}, \varrho' \frac{dv}{dt}, \varrho' \frac{dw}{dt}$ and for m we have to introduce the value 2. We see that this is true for 2a, 2b, 2c, 3d, 3e and 3f; they thus become the well-known equations

$$\sigma_x = -p - \frac{2}{3}\eta\, a \frac{\partial \theta}{\partial t} + 2\eta \frac{\partial u}{\partial x} \quad . \quad (8a) \qquad \tau_x = \eta\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \quad . \quad (8d)$$

$$\sigma_y = -p - \frac{2}{3}\eta\, a \frac{\partial \theta}{\partial t} + 2\eta \frac{\partial v}{\partial y} \quad . \quad (8b) \qquad \tau_y = \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad . \quad (8e)$$

$$\sigma_z = -p - \frac{2}{3}\eta\, a \frac{\partial \theta}{\partial t} + 2\eta \frac{\partial w}{\partial z} \quad . \quad (8c) \qquad \tau_z = \eta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad . \quad (8f)$$

where p denotes the pressure and, therefore, equals—1/3 σ and η indicates the coefficient of dynamic viscosity. In 8a, 8b, and 8c the stresses have been brought to the left side and the velocity terms to the right for getting them in the same shape as the equations 3a, 3b and 3c which themselves become indeterminate. The equation 2d simplifies to

$$e = a \frac{\partial \theta}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where e now is the divergence of the velocity.

Introducing the formulae 8 in the equations 1 we obtain the conditions for equilibrium, corresponding to the equations 4, in the shape

$$\varrho' \frac{du}{dt} = P_x - \frac{\partial p}{\partial x} + \eta\, \nabla^2 u + \frac{1}{3}\eta\, a \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial t}\right) \quad . \quad . \quad . \quad . \quad (10a)$$

$$\varrho' \frac{dv}{dt} = P_y - \frac{\partial p}{\partial y} + \eta\, \nabla^2 v + \frac{1}{3}\eta\, a \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial t}\right) \quad . \quad . \quad . \quad . \quad (10b)$$

$$\varrho' \frac{dw}{dt} = P_z - \frac{\partial p}{\partial z} + \eta\, \nabla^2 w + \frac{1}{3}\eta\, a \frac{\partial}{\partial z}\left(\frac{\partial \theta}{\partial t}\right) + a\,\varrho' g\,\theta \quad . \quad (10c)$$

Because of the terms of D'ALEMBERT they here represent the equations of motion.

The equation 5 loses its value and 6 becomes

$$\nabla^2 p = \nabla P - \varrho' a \frac{d}{dt}\left(\frac{\partial \theta}{\partial t}\right) + \frac{4}{3}\eta\, a\, \nabla^2\left(\frac{\partial \theta}{\partial t}\right) + a\,\varrho' g \frac{\partial \theta}{\partial z} \quad . \quad (11)$$

This equation can be checked by differentiating 4a with regard to x, 4b with regard to y and 4c with regard to z and by adding the results.

The equations 7 remain valid but as the addition of the terms of

D'ALEMBERT to the mass-forces P brings about the reintroduction of the velocity components they loose their value as formulae where only the stresses occur. We shall, therefore, refrain from writing them down.

We shall also refrain from further developing the above formulae by replacing in the well-known way the differentiation d/dt referring to the change for an element of the fluid by the differentiation $\partial/\partial t$ referring to the change at a fixed point in space; for slow motion problems they may be interchanged. Our purpose in writing down the above equations was only to show that indeed the above transformation changes the equations for an elastic solid into those for a viscous fluid.

§ 2. *Solution of the equations of § 1 for the general problem that P_z, w, σ_z and θ are functions of z multiplied by the same function K of x and y determined by*

$$\nabla^2 K + f^2 K = 0 \quad \ldots \ldots \ldots \quad (12)$$

while the resultants of P_x and P_y, of u and v and of τ_y and τ_x are functions of z multiplied by ∇K.

We begin by dealing with the problem for an elastic solid, i.e. for an arbitrary value of m. We assume

$$\left.\begin{aligned} P_x &= P_1 \frac{\partial K}{\partial x} & u &= v_1 \frac{\partial K}{\partial x} & \theta &= \theta_0 K \\ P_y &= P_1 \frac{\partial K}{\partial \varphi} & v &= v_1 \frac{\partial K}{\partial \varphi} & & \\ P_z &= P_0 K & w &= w_0 K & & \end{aligned}\right\} \quad \ldots \quad (13)$$

P_1, P_0, v_1, w_0 and θ_0 are functions of z. The function K of x and y is given by the general solution of 12:

$$K = C_1 e^{ilx} e^{isy} + C_2 e^{-ilx} e^{isy} \quad \ldots \ldots \quad (14a)$$

where

$$l^2 + s^2 = f^2 \quad \ldots \ldots \ldots \quad (14b)$$

and C_1 and C_2 two integration-constants. l and s may be imaginary. Any sum of solutions for different values of l and s of course also fulfills the conditions.

If l and s are both real and if we choose the X and Y axis in a suitable way we can write the solution in the shape

$$K = C \cos (l + \varphi_x) \cos (s y + \varphi_y) \quad \ldots \ldots \quad (15)$$

If l is imaginary we can put $k = i\,l$ and we find if we again choose the X and Y axis appropriately

$$K = (A\, e^{-kx} + B\, e^{kx}) \cos (s y + \varphi_y) \quad \ldots \ldots \quad (16a)$$

with

$$s^2 - k^2 = f^2. \quad \ldots \ldots \ldots \quad (16b)$$

In the first shape the function is periodic in the sense of the two coordinateaxes and in the second shape only in the sense of the Y axis. So the formulae of this § can be applied to problems of which the conditioning quantities can be developed in Fourier-series in one or in two directions. Applying 13 we find

$$e = e_0 K \text{ with } e_0 = -f^2 v_1 + \frac{\partial w_0}{\partial z} \quad \dots \dots \quad (17)$$

and introducing 13 in 4c

$$\frac{\partial^2 w_0}{\partial z^2} - f^2 w_0 + \frac{m}{m-2}\frac{\partial e_0}{\partial z} + \frac{P_0}{G} - \frac{2(m+1)}{3(m-2)}\alpha\frac{\partial \theta_0}{\partial z} + \frac{\alpha \varrho' g}{G}\theta_0 = 0 \quad (18a)$$

The equations 4*a* and 4*b* give the same result

$$\frac{\partial^2 v_1}{\partial z^2} - f^2 v_1 + \frac{m}{m-2}e_0 + \frac{P_1}{G} - \frac{2(m+1)}{3(m-2)}\alpha\theta_0 = 0 \quad \dots \quad (18b)$$

and using 17 for eliminating v_1 we obtain

$$\left.\begin{aligned} &\frac{\partial^3 w_0}{\partial z^3} - f^2\frac{\partial w_0}{\partial z} + \frac{2(m-1)}{m-2}f^2 e_0 - \frac{\partial^2 e_0}{\partial z^2} + \frac{f^2}{G}P_1 - \\ &\qquad\qquad - \frac{2(m+1)}{3(m-2)}f^2\alpha\theta_0 = 0 \end{aligned}\right\} \quad \dots \quad (18b')$$

From 18*a* and 18*b'* we can eliminate e_0 and dividing the result by $2(m-1)/m$ we get the equation for w_0

$$\frac{\partial^4 w_0}{\partial z^4} - 2f^2\frac{\partial^2 w_0}{\partial z^2} + f^4 w_0 + F = 0 \quad \dots \dots \quad (19a)$$

with

$$\left.\begin{aligned} F = &\frac{1}{G}\left[-f^2 P_0 + \frac{(m-2)}{2(m-1)}\frac{\partial^2 P_0}{\partial z^2} + \frac{m}{2(m-1)}f^2\frac{\partial P_1}{\partial z}\right] + \\ &+ \frac{(m+1)}{3(m-1)}\alpha\left[f^2\frac{\partial\theta_0}{\partial z} - \frac{\partial^3\theta_0}{\partial z^3}\right] + \frac{\alpha\varrho' g}{G}\left[-f^2\theta_0 + \frac{(m-2)}{2(m-1)}\frac{\partial^2\theta_0}{\partial z^2}\right]. \end{aligned}\right\} \quad (19b)$$

This is the fundamental equation for our problem and it is interesting to notice that the terms of 19*a* containing w_0 do not depend on the value of m; this value only affects the function F which comprises the effects of the mass-forces, of the temperature expansion (second term) and the temperature gravity effect (third term).

After deducing w_0 from 19*a* we can derive e_0 from 18*a*:

$$\left.\begin{aligned} \frac{\partial e_0}{\partial z} = &\frac{(m-2)}{m}f^2 w_0 - \frac{(m-2)}{m}\frac{\partial^2 w_0}{\partial z^2} + \frac{2(m+1)}{3m}\alpha\frac{\partial\theta_0}{\partial z} - \\ &\qquad - \frac{(m-2)}{m}\frac{P_0}{G} - \frac{\alpha\varrho' g}{G}\frac{(m-2)}{m}\theta_0 \end{aligned}\right\} \quad . \quad (20)$$

and v_1 from 17:

$$v_1 = \frac{1}{f^2}\left(\frac{\partial w_0}{\partial z} - e_0\right) \quad \ldots \ldots \ldots \quad (21)$$

We can further deduce the stresses from the equations 3 in combination with 12 and 13 and we find

$$\frac{\sigma_x}{2\,G} = \left[\frac{e_0}{m-2} - \frac{(m+1)}{3\,(m-2)}\,\alpha\,\theta_1\right] K + v_0 \frac{\partial^2 K}{\partial x^2} \quad \ldots \quad (22a)$$

$$\frac{\sigma_y}{2\,G} = \left[\frac{e_0}{m-2} - \frac{(m+1)}{3\,(m-2)}\,\alpha\,\theta_1\right] K + v_0 \frac{\partial^2 K}{\partial y^2} \quad \ldots \quad (22b)$$

$$\frac{\sigma_z}{2\,G} = \left[\frac{\partial w_0}{\partial z} + \frac{e_0}{m-2} - \frac{(m+1)}{3(m-2)}\,\alpha\,\theta_0\right] K \quad \ldots \quad (22c)$$

$$\frac{\tau_x}{2\,G} = \frac{1}{2}\left(w_0 + \frac{\partial v_1}{\partial z}\right)\frac{\partial K}{\partial y} \quad \ldots \ldots \quad (22d)$$

$$\frac{\tau_y}{2\,G} = \frac{1}{2}\left(w_0 + \frac{\partial v_1}{\partial z}\right)\frac{\partial K}{\partial x} \quad \ldots \ldots \quad (22e)$$

$$\frac{\tau_z}{2\,G} = v_1 \frac{\partial^2 K}{\partial x\,\partial y} \quad \ldots \ldots \ldots \quad (22f)$$

From the general solution 14 of K we derive

$$\frac{\partial^2 K}{\partial x^2} = -\,l^2 K \qquad\qquad \frac{\partial^2 K}{\partial y^2} = -\,s^2 K \quad \ldots \quad (23)$$

and so we can replace the formulae 22a and 22b by

$$\frac{\sigma_x}{2\,G} = \left[-\,l^2 v_1 + \frac{e_0}{m-2} - \frac{(m+1)}{3(m-2)}\,\alpha\,\theta_0\right] K \quad \ldots \quad (22a')$$

$$\frac{\sigma_y}{2\,G} = \left[-\,s^2 v_1 + \frac{e_0}{m-2} - \frac{(m+1)}{3\,(m-2)}\,\alpha\,\theta_0\right] K \quad \ldots \quad (22b')$$

and for the sum $\sigma = \sigma_x + \sigma_y + \sigma_z$ we find

$$\frac{\sigma}{2\,G} = \frac{m+1}{m-2}\left[-\,f^2 v_1 + \frac{\partial w_0}{\partial z} - \alpha\,\theta_0\right] K \quad \ldots \quad (24)$$

which is in harmony with the equations 2d and 17.

We see that our formulae fulfill all the conditions and so they constitute a valid solution of the case presented by the assumptions 13.

As we have discussed in § 1 we can obtain the formulae for viscous fluids of which the elastic deformations etc. are neglected by putting $m = 2$, by adding the D'ALEMBERT terms to the mass-forces and by substituting $\frac{\partial \theta}{\partial t}$ to θ in the temperature expansion terms. We thus get

$$\frac{\partial^4 w_0}{\partial z^4} - 2\,f^2 \frac{\partial^2 w_0}{\partial z^2} + f^4 w_0 + F = 0 \quad \ldots \ldots \quad (25a)$$

$$F = \frac{f^2 \varrho'}{\eta}\left[\frac{d w_0}{d t} - \frac{d}{d t}\left(\frac{\partial v_1}{\partial z}\right)\right] - \frac{f^2}{\eta}\left[P_0 - \frac{\partial P_1}{\partial z}\right] + \left.\begin{matrix} \\ + a\left[f^2 \frac{\partial^2 \theta_0}{\partial z\, \partial t} - \frac{\partial^4 \theta_0}{\partial z^3 \partial t}\right] - \frac{a \varrho' g}{\eta} f^2 \theta_0 \end{matrix}\right\} \quad . \quad (25b)$$

$$v_1 = \frac{1}{f^2}\left(\frac{\partial w_0}{\partial z} - a \frac{\partial \theta}{\partial t}\right) \quad . \ . \ . \ . \ . \ . \ . \quad (26)$$

$$e_0 = a \frac{\partial \theta}{\partial t} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (27)$$

By introducing 13 and 23 in 8 and putting $p = p_0 K$ we obtain for the stresses

$$\frac{\sigma_x}{2\eta} = -\left[\frac{p_0}{2\mu} + \frac{1}{3} a \frac{\partial \theta_0}{\partial t} + v_1 l^2\right] K . \quad . \ . \ . \ . \quad (28a)$$

$$\frac{\sigma_y}{2\eta} = -\left[\frac{p_0}{2\eta} + \frac{1}{3} a \frac{\partial \theta_0}{\partial t} + v_1 m^2\right] K \quad . \ . \ . \ . \quad (28b)$$

$$\frac{\sigma_z}{2\eta} = -\left[\frac{p_0}{2\eta} + \frac{1}{3} a \frac{\partial \theta_0}{\partial t} - \frac{\partial w_0}{\partial z}\right] K . \quad . \ . \ . \ . \quad (28c)$$

$$\frac{\tau_x}{2\eta} = \frac{1}{2}\left(w_0 + \frac{\partial v_1}{\partial z}\right)\frac{\partial K}{\partial y} \quad . \ . \ . \ . \ . \ . \quad (29d)$$

$$\frac{\tau_y}{2\eta} = \frac{1}{2}\left(w_0 + \frac{\partial v_1}{\partial z}\right)\frac{\partial K}{\partial x} . \quad . \ . \ . \ . \ . \ . \ . \quad (28e)$$

$$\frac{\tau_z}{2\eta} = v_1 \frac{\partial^2 K}{\partial x\, \partial y} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (28f)$$

Lastly we can derive from 10*a* or 10*b* in combination with 12 and 13

$$\frac{p_0}{\eta} = \frac{P_1}{\eta} - \frac{\varrho'}{\eta}\frac{d v_1}{\partial t} + \frac{\partial^2 v_1}{\partial z^2} - f^2 v_1 + \frac{1}{3} a \frac{\partial \theta_0}{\partial t} . \quad . \ . \ . \quad (29)$$

As the function F of formulae 25*a* and 25*b* still contains dw_0/dt and $d(\partial v_0/\partial z)/dt$ we shall in general have to develop these formulae before we can use the solution. In one case, however, we can apply them as they are, i.e. for slow steady motion problems. In that case d/dt and $\partial/\partial t$ are zero and so we get:

$$F = -\frac{f^2}{\eta}\left(P_0 - \frac{\partial P_1}{\partial z}\right) - \frac{a \varrho' g}{\eta} f^2 \theta_0 \quad . \ . \ . \ . \ . \ . \quad (30)$$

$$v_1 = \frac{1}{f^2}\frac{\partial w_0}{\partial z} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (31)$$

$$\frac{\sigma_x}{2\eta} = -\left[\frac{p_0}{2\eta} + v_1 l^2\right] K \quad (32a) \qquad \frac{\tau_x}{2\eta} = \frac{1}{2}\left(w_0 + \frac{\partial v_1}{\partial z}\right)\frac{\partial K}{\partial y} \quad (32d)$$

$$\frac{\sigma_y}{2\eta} = -\left[\frac{p_0}{2\eta} + v_1 m^2\right] K \quad (32b) \qquad \frac{\tau_y}{2\eta} = \frac{1}{2}\left(w_0 + \frac{\partial v_1}{\partial z}\right)\frac{\partial K}{\partial x} \quad (32e)$$

$$\frac{\sigma_z}{2\eta} = -\left[\frac{p_0}{2\eta} - \frac{\partial w_0}{\partial z}\right] K \quad (32c) \qquad \frac{\tau_z}{2\eta} = v_1 \frac{\partial^2 K}{\partial x \partial y} \quad (32f)$$

$$\frac{\varphi_0}{\eta} = \frac{P_1}{\eta} + \frac{\partial^2 v_1}{\partial z^2} - f^2 v_1 \quad (33)$$

These formulae may e.g. be used for problems of slow steady convection. The further development of the solution depends on the conditions determining the distribution of the temperature.

§ 3. *Equations for elastic solids in spherical coordinates ϱ, δ, λ; over the area investigated gravity, working in the sense of radius ϱ, is assumed to be constant; the temperature may be variable.*

We shall again begin by dealing with the problem for an elastic solid. As coordinates of a point we take as usual the radius ϱ, the angle δ between the radius and the polar axis and the angle λ between the meridian plane in which δ is measured and a fixed plane through the polar axis. Applying this system to the Earth we shall measure δ from the North pole and we shall give positive sign to λ if it is measured westwards. For the shearing stresses τ_ϱ, τ_δ, τ_λ we shall fix the sign in such a way that the components τ_ϱ resp. τ_δ working in the meridian plane $\lambda + d\lambda$ on an element comprised between that plane and the meridian plane λ have positive signs in the sense of increasing δ resp. increasing ϱ, and that the component τ_λ working in the sphere with radius $\varrho + d\varrho$ on an element comprised between that sphere and another with radius ϱ has positive sign in the sense of increasing δ.

For the conditions of equilibrium we find
in the sense of ϱ

$$\frac{2\sigma_\varrho - \sigma_\delta - \sigma_\lambda}{\varrho} + \frac{\partial \sigma_\varrho}{\partial \varrho} + \frac{1}{\varrho \sin\delta}\frac{\partial \tau_\delta}{\partial \lambda} + \frac{1}{\varrho}\frac{\partial \tau_\lambda}{\partial \delta} + \frac{\tau_\lambda}{\varrho}\operatorname{cotg}\delta + P_\varrho + \alpha\varrho' g \theta = 0, \quad (34a)$$

in the sense of δ

$$\frac{(\sigma_\delta - \sigma_\lambda)}{\varrho}\operatorname{cotg}\delta + \frac{1}{\varrho}\frac{\partial \sigma_\delta}{\partial \delta} + \frac{1}{\varrho \sin\delta}\frac{\partial \tau_\varrho}{\partial \lambda} + 3\frac{\tau_\lambda}{\varrho} + \frac{\partial \tau_\lambda}{\partial \varrho} + P_\delta = 0. \quad (34b)$$

in the sense of λ

$$\frac{1}{\varrho \sin\delta}\frac{\partial \sigma_\lambda}{\partial \lambda} + 2\frac{\tau_\varrho}{\varrho}\operatorname{cotg}\delta + \frac{1}{\varrho}\frac{\partial \tau_\varrho}{\partial \delta} + 3\frac{\tau_\delta}{\varrho} + \frac{\partial \tau_\delta}{\partial \varrho} + P_\lambda = 0. \quad (34c)$$

The relation of the elastical displacements (V_ϱ, V_δ, V_λ) to the stresses is given

in the sense of ϱ by

$$\frac{\partial V_\varrho}{\partial \varrho} = \frac{1}{E}\left(\sigma_\varrho - \frac{\sigma_\delta + \sigma_\lambda}{m}\right) + \frac{a}{3}\,\theta = \frac{1}{2G}\left(\sigma_\varrho - \frac{\sigma}{m+1}\right) + \frac{a}{3}\,\theta, \quad (35a)$$

in the sense of δ by

$$\frac{1}{\varrho}\frac{\partial V_\delta}{\partial \delta} + \frac{V_\varrho}{\varrho} = \frac{1}{E}\left(\sigma_\delta - \frac{\sigma_\varrho + \sigma_\lambda}{m}\right) + \frac{a}{3}\,\theta = \frac{1}{2G}\left(\sigma_\delta - \frac{\sigma}{m+1}\right) + \frac{a}{3}\theta, \quad (35b)$$

in the sense of λ by

$$\left.\begin{aligned} \frac{1}{\varrho \sin\delta}\frac{\partial V_\lambda}{\partial \lambda} + \frac{V_\varrho}{\varrho} + \operatorname{cotg}\delta\,\frac{V_\delta}{\varrho} &= \frac{1}{E}\left(\sigma_\lambda - \frac{\sigma_\varrho + \sigma_\delta}{m}\right) + \frac{a}{3}\theta = \\ &= \frac{1}{2G}\left(\sigma_\lambda - \frac{\sigma}{m+1}\right) + \frac{a}{3}\theta, \end{aligned}\right\} \quad (35c)$$

in which again

$$\sigma = \sigma_\varrho + \sigma_\delta + \sigma_\lambda.$$

From these equations we may derive

$$\left.\begin{aligned} &\frac{\partial V_\varrho}{\partial \varrho} + 2\,\frac{V_\varrho}{\varrho} + \frac{1}{\varrho}\frac{\partial V_\delta}{\partial \delta} + \operatorname{cotg}\delta\,\frac{V_\delta}{\varrho} + \\ &+ \frac{1}{\varrho \sin\delta}\frac{\partial V_\lambda}{\partial \lambda} = e = \frac{m-2}{2(m+1)}\frac{\sigma}{G} + a\theta \end{aligned}\right\} \quad (36)$$

From the equations 35 we further can deduce the expression of the stresses in the displacements

$$\frac{\sigma_\varrho}{2G} = \frac{\partial V_\varrho}{\partial \varrho} + \frac{e}{m-2} - \frac{m+1}{3(m-2)}\,a\theta \quad (37a)$$

$$\frac{\sigma_\delta}{2G} = \frac{1}{\varrho}\frac{\partial V_\delta}{\partial \delta} + \frac{V_\varrho}{\varrho} + \frac{e}{m-2} - \frac{m+1}{3(m-2)}\,a\theta \quad (37b)$$

$$\frac{\sigma_\lambda}{2G} = \frac{1}{\varrho \sin\delta}\frac{\partial V_\lambda}{\partial \lambda} + \frac{V_\varrho}{\varrho} + \operatorname{cotg}\delta\,\frac{V_\delta}{\varrho} + \frac{e}{m-2} - \frac{m+1}{3(m-2)}\,a\theta \quad (37c)$$

$$\frac{\tau_\varrho}{G} = \frac{1}{\varrho \sin\delta}\frac{\partial V_\delta}{\partial \lambda} + \frac{1}{\varrho}\frac{\partial V_\lambda}{\partial \delta} - \operatorname{cotg}\delta\,\frac{V_\lambda}{\varrho} \quad (37d)$$

$$\frac{\tau_\delta}{G} = \frac{\partial V_\lambda}{\partial \varrho} + \frac{1}{\varrho \sin\delta}\frac{\partial V_\varrho}{\partial \lambda} - \frac{V_\lambda}{\varrho} \quad (37e)$$

$$\frac{\tau_\lambda}{G} = \frac{\partial V_\delta}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial V_\varrho}{\partial \delta} - \frac{V_\delta}{\varrho} \quad (37f)$$

Introduced in 34 we obtain the conditions of equilibrium expressed in the elastic displacements. Taking account of

$$\nabla^2 = \frac{\partial^2}{\partial \varrho^2} + \frac{2}{\varrho}\frac{\partial}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial \delta^2} + \frac{1}{\varrho^2 \sin^2\delta}\frac{\partial^2}{\partial \lambda^2} + \frac{\operatorname{cotg}\delta}{\varrho^2}\frac{\partial}{\partial \delta} \quad (38)$$

we find

$$\left.\begin{aligned}\nabla^2 V_\varrho - \frac{2}{\varrho^2} V_\varrho - \frac{2}{\varrho^2}\frac{\partial V_\delta}{\partial \delta} - \frac{2 \operatorname{cotg} \delta}{\varrho^2} V_\delta - \frac{2}{\varrho^2 \sin\delta}\frac{\partial V_\lambda}{\partial \lambda} + \\ + \frac{m}{m-2}\frac{\partial e}{\partial \varrho} - \frac{2(m+1)}{3(m-2)} a \frac{\partial \theta}{\partial \varrho} + \frac{P_\varrho}{G} + \frac{a \varrho' g}{G}\theta = 0\end{aligned}\right\} \quad (39a)$$

$$\left.\begin{aligned}\nabla^2 V_\delta + \frac{2}{\varrho^2}\frac{\partial V_\varrho}{\partial \delta} - \frac{1}{\varrho^2 \sin^2\delta} V_\delta - \frac{2\cos\delta}{\varrho^2\sin^2\delta}\frac{\partial V_\lambda}{\partial \lambda} + \\ + \frac{m}{m-2}\frac{1}{\varrho}\frac{\partial e}{\partial \delta} - \frac{2(m+1)}{3(m-2)}\frac{a}{\varrho}\frac{\partial \theta}{\partial \delta} + \frac{P_\delta}{G} = 0\end{aligned}\right\} \quad . \; . \; . \quad (39b)$$

$$\left.\begin{aligned}\nabla^2 V_\lambda + \frac{2}{\varrho^2\sin\delta}\frac{\partial V_\varrho}{\partial \lambda} + \frac{2\cos\delta}{\varrho^2\sin^2\delta}\frac{\partial V_\delta}{\partial \lambda} - \frac{1}{\varrho^2\sin^2\delta} V_\lambda + \\ + \frac{m}{m-2}\frac{1}{\varrho\sin\delta}\frac{\partial e}{\partial \lambda} - \frac{2(m+1)}{3(m-2)}\frac{a}{\varrho\sin\delta}\frac{\partial \theta}{\partial \lambda} + \frac{P_\lambda}{G} = 0\end{aligned}\right\} \quad . \quad (39c)$$

As we shall not require them for the problem of the next § we shall abstain from deducing the comptability conditions in our coordinates.

We shall only examine the case of viscous fluids for the problem of the next §.

§ 4. *Solution of the equations of § 3 for the general problem that P_ϱ, V_ϱ, σ_ϱ and θ are functions of ϱ multiplied by the same spherical harmonic K_n and that the resultants of P_δ and P_λ, of V_δ and V_λ and of τ_λ and τ_δ are functions of ϱ multiplied by ∇K_n.*

We assume

$$\begin{aligned} &P_\varrho = P_0 K_n & &V_\varrho = w_0 K_n & &\theta = \theta_0 K_n \qquad (40)\\ &P_\delta = P \frac{\partial K_n}{\partial \delta} & &V_\delta = v_0 \frac{\partial K_n}{\partial \delta} & & \\ &P_\lambda = \frac{P}{\sin\delta}\frac{\partial K_n}{\partial \lambda} & &V_\lambda = \frac{v_0}{\sin\delta}\frac{\partial K_n}{\partial \lambda} & & \end{aligned}$$

The quantities P_0, P, w_0, v_0 and θ_0 are functions of ϱ. Introducing 40 in 36 and using 38 we find

$$e = \left(\frac{\partial w_0}{\partial \varrho} + \frac{2}{\varrho} w_0\right) K_n + \varrho\, v_0 \nabla^2 K_n$$

As for all spherical harmonics

$$\nabla^2 (\varrho^n K_n) = 0$$

we get by means of 38

$$\left[\left(\frac{\partial^2}{\partial \varrho^2} + \frac{2}{\varrho}\frac{\partial}{\partial \varrho}\right)\varrho^n\right] K_n + \varrho^n \nabla^2 K_n = 0$$

or

$$n(n+1)\,\varrho^{n-2}K_n + \varrho^n \nabla^2 K_n = 0$$

and so we obtain the important relation

$$\nabla^2 K_n = -\frac{n(n+1)}{\varrho^2} K_n. \quad \ldots \ldots \ldots \quad (41)$$

Using this we get

$$e_0 = \frac{\partial w_0}{\partial \varrho} + \frac{2}{\varrho} w_0 - \frac{n(n+1)}{\varrho} v_0 \quad \ldots \ldots \ldots \quad (42)$$

Substituting 40 in 39a we obtain

$$\left.\begin{aligned} & n(n+1)\left[\frac{1}{\varrho}\frac{\partial v_0}{\partial \varrho} + \frac{v_0}{\varrho^2} - \frac{w_0}{\varrho^2}\right] + 2\frac{(m-1)}{(m-2)}\frac{\partial e_0}{\partial \varrho} - \\ & \qquad - \frac{2(m+1)}{3(m-2)}\alpha\frac{\partial \theta_0}{\partial \varrho} + \frac{P_0}{G} + \frac{\alpha \varrho' g}{G}\theta_0 = 0 \end{aligned}\right\} \quad \ldots \quad (43a)$$

and in 39*b* or 39*c*

$$-\frac{1}{\varrho}\frac{\partial w_0}{\partial \varrho} + \frac{2}{\varrho}\frac{\partial v_0}{\partial \varrho} + \frac{\partial^2 v_0}{\partial \varrho^2} + 2\frac{(m-1)}{(m-2)}\frac{e_0}{\varrho} - \frac{2(m+1)}{3(m-2)}\frac{\alpha}{\varrho}\theta_0 + \frac{P}{G} = 0 \quad . \quad (43b)$$

Eliminating e_0 and v_0 from 42, 43a and 43*b* we find the fundamental equation for our problem

$$\left.\begin{aligned} & \frac{\partial^4 w_0}{\partial \varrho^4} + \frac{8}{\varrho}\frac{\partial^3 w_0}{\partial \varrho^3} - \frac{2(n^2+n-6)}{\varrho^2}\frac{\partial^2 w_0}{\partial \varrho^2} - \frac{4n(n+1)}{\varrho^3}\frac{\partial w_0}{\partial \varrho} + \\ & \qquad + (n-1)\,n\,(n+1)(n+2)\frac{w_0}{\varrho^4} + F = 0 \end{aligned}\right\} \quad \ldots \quad (44a)$$

where

$$\left.\begin{aligned} F = \frac{1}{G}\Bigg[& \frac{(4-m)}{2(m-1)}\frac{n(n+1)}{\varrho^2}P + \frac{m}{2(m-1)}\frac{n(n+1)}{\varrho}\frac{\partial P}{\partial \varrho} - \\ & - \left\{n(n+1) - \frac{3(m-2)}{(m-1)}\right\}\frac{P_0}{\varrho^2} + \frac{3(m-2)}{(m-1)\varrho}\frac{\partial P_0}{\partial \varrho} \frac{(m-2)}{2(m-1)}\frac{\partial^2 P_0}{\partial \varrho^2}\Bigg] + \\ & + \frac{(m+1)}{3(m-1)}\alpha\left[\frac{2n(n+1)}{\varrho^3}\theta_0 + \frac{(n^2+n-6)}{\varrho^2}\frac{\partial \theta_0}{\partial \varrho} - \frac{6}{\varrho}\frac{\partial^2 \theta_0}{\partial \varrho^2} - \frac{\partial^3 \theta_0}{\partial \varrho^3}\right] + \\ & \qquad + \frac{\alpha \varrho' g}{G}\Bigg[-\left\{n(n+1) - \frac{3(m-2)}{(m-1)}\right\}\frac{\theta_0}{\varrho^2} + \\ & \qquad\qquad + \frac{3(m-2)}{(m-1)\varrho}\frac{\partial \theta_0}{\partial \varrho} + \frac{(m-2)}{2(m-1)}\frac{\partial^2 \theta_0}{\partial \varrho^2}\Bigg] \end{aligned}\right\} \quad . \quad (44b)$$

As we could expect to be the case after the result found in formula 19 the terms of 44a containing w_0 do not depend on the value of m; this quantity only occurs in the function F. The first term of this function again

represents the effects of the mass-forces, the second that of the temperature expansion and the third the gravitational effect brought about by the temperature density change.

After deriving w_0 from 19 we can find e_0 by means of the following formula obtained by eliminating v_0 from 42 and 43a

$$\left.\begin{aligned} 2e_0 - \frac{m}{m-2}\varrho\frac{\partial e_0}{\partial \varrho} = \varrho\frac{\partial^2 w_0}{\partial \varrho^2} + 4\frac{\partial w_0}{\partial \varrho} - \\ -\frac{(n^2+n-2)}{\varrho}w_0 - \frac{2(m+1)}{3(m-2)}\alpha\varrho\frac{\partial\theta}{\partial\varrho} + \varrho\frac{P_0}{G} + \frac{\alpha\varrho' g}{G}\varrho\theta_0 \end{aligned}\right\} \quad (45)$$

It is easy then to deduce v_0 from 42:

$$v_0 = \frac{1}{n(n+1)}\left[2w_0 + \varrho\frac{\partial w_0}{\partial\varrho} - \varrho e_0\right] \quad . \quad . \quad . \quad . \quad (46)$$

By introducing 40 in the formulae 37 we find for the stresses

$$\frac{\sigma_\varrho}{2G} = \left[\frac{\partial w_0}{\partial\varrho} + \frac{e_0}{m-2} - \frac{m+1}{3(m-2)}\alpha\theta_0\right]K_n \quad . \quad . \quad . \quad . \quad (47a)$$

$$\frac{\sigma_\delta}{2G} = \left[\frac{w_0}{\varrho} + \frac{e}{m-2} - \frac{m+1}{3(m-2)}\alpha\theta_0\right]K_n + \frac{v_0}{\varrho}\frac{\partial^2 K_n}{\partial\delta^2} \quad . \quad . \quad . \quad . \quad (47b)$$

$$\frac{\sigma_\lambda}{2G} = \left[\frac{w_0}{\varrho} + \frac{e}{m-2} - \frac{m+1}{3(m-2)}\alpha\theta_0\right]K_n + \frac{v_0}{\varrho}\left[\varrho^2\nabla^2 K_n - \frac{\partial^2 K_n}{\partial\delta^2}\right] \quad (47c)$$

$$\frac{\tau_\varrho}{2G} = \frac{v_0}{\varrho\sin\delta}\left[\frac{\partial^2 K_n}{\partial\delta\,\partial\lambda} - \operatorname{cotg}\delta\frac{\partial K_n}{\partial\lambda}\right] \quad . \quad . \quad . \quad . \quad (47d)$$

$$\frac{\tau_\delta}{2G} = \frac{1}{2\sin\delta}\left[\frac{\partial v_0}{\partial\varrho} + \frac{w_0 - v_0}{\varrho}\right]\frac{\partial K_n}{\partial\lambda} \quad . \quad . \quad . \quad . \quad (47e)$$

$$\frac{\tau_\lambda}{2G} = \frac{1}{2}\left[\frac{\partial v_0}{\partial\varrho} + \frac{w_0 - v_0}{\varrho}\right]\frac{\partial K_n}{\partial\delta} \quad . \quad . \quad . \quad . \quad (47f)$$

and for the sum $\sigma = \sigma_\varrho + \sigma_\delta + \sigma_\lambda$ we derive from 2d and 42

$$\frac{\sigma}{2G} = \frac{m+1}{m-2}\left[\frac{\partial w_0}{\partial\varrho} + 2\frac{w_0}{\varrho} - n(n+1)\frac{v_0}{\varrho} - \alpha\theta_0\right] \quad . \quad . \quad (48)$$

As our formulae fulfill all the conditions we may conclude that they indeed give the solution for the case assumed by the suppositions 40. These suppositions thus prove to allow a valid solution.

Examining again the problem of § 2 we see that it may be considered as the limit to which our present problem tends when ϱ and n become infinite. In that case the equation 12 can be identified with 41 if we put

$$f^2 = \lim_{\substack{n=\infty \\ \varrho_m=\infty}}\left(\frac{n(n+1)}{\varrho_m^2}\right) \quad . \quad . \quad . \quad . \quad (49)$$

in which ϱ_m is a mean value of ϱ over the area considered. All the formulae of this § then change into those of § 2. The quantities w_0, P_0, θ_0 keep their meaning, while v_1 and P_1 of § 2 correspond to ϱv_0 and ϱP of this §. The factor ϱ in these last quantities arises from the fact that ∇K, of which the components $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial y}$ appear in 13, corresponds to ∇K_n with the components $\frac{1}{\varrho}\frac{\partial Kn}{\partial \delta}$ and $\frac{1}{\varrho \sin \delta}\frac{\partial Kn}{\partial \lambda}$ and that the formulae 40 have included the factors $1/\varrho$ of these last differential-quotients in the quantities v_1 resp. P_1.

The meaning of 49 can be illustrated by introducing the length L given by $L = 2\pi/\mathfrak{f}$. For the special case of the problem of § 2 that $m = 0$ the function K becomes constant in the direction of y and only remains periodic in the direction of x; L is then the wave-length in that sense. From 49 we conclude

$$L = \lim_{\substack{n=\infty \\ \varrho_m=\infty}} \left(\frac{2\pi\varrho_m}{n}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

We shall now consider the problem of this § for viscous fluids and we shall again neglect the elastic deformations and the variations of density, except in so far as they modify the action of gravity. As we have mentioned in § 1 we can derive the formulae for this case by introducing $m = 2$, by adding the D'ALEMBERT terms to the mass-forces and by substituting $\frac{\partial \theta}{\partial t}$ to θ in the temperature expansion terms. We shall not write down here the formulae thus obtained for the general case, but we shall limit us to the case of slow steady motion already mentioned at the close of § 2. In that case we may neglect the D'ALEMBERT terms and as we have $\frac{\partial \theta}{\partial t} = 0$ the temperature expansion terms disappear also. We thus obtain

$$\left.\begin{aligned} &\frac{\partial^4 w_0}{\partial \varrho^4} + \frac{8}{\varrho}\frac{\partial^3 w_0}{\partial \varrho^3} - \frac{2(n^2+n-6)}{\varrho^2}\frac{\partial^2 w_0}{\partial \varrho^2} - \frac{4n(n+1)}{\varrho^3}\frac{\partial w_0}{\partial \varrho} + \\ &+ (n-1)\,n\,(n+1)(n+2)\frac{w_0}{\varrho^4} + \frac{n(n+1)}{\eta}\left[\frac{1}{\varrho}\frac{\partial P}{\partial \varrho} + \frac{P-P_0}{\varrho^2}\right] - \\ &\qquad - \frac{n(n+1)}{\varrho^2}\frac{\alpha\varrho' g}{\eta}\,\theta_0 = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (51)$$

and

$$v_0 = \frac{1}{n(n+1)}\left(2w_0 + \varrho\frac{\partial w_0}{\partial \varrho}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Denoting again the pressure $-\sigma/3$ by $p = p_0 K_n$ and making use of

$$\frac{e - \alpha\theta}{m-2} = -\frac{p}{2\eta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

derived from $2d$, we deduce from the formulae 47 for the stresses

$$\frac{\sigma_\varrho}{2\eta} = \left[-\frac{p_0}{2\eta} + \frac{\partial w_0}{\partial \varrho} \right] K_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54a)$$

$$\frac{\sigma_\delta}{2\eta} = \left[-\frac{p_0}{2\eta} + \frac{w_0}{\varrho} \right] K_n + \frac{v_0}{\varrho} \frac{\partial^2 K_n}{\partial \delta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54b)$$

$$\frac{\sigma_\lambda}{2\eta} = \left[-\frac{p_0}{2\eta} + \frac{w_0}{\varrho} \right] K_n + \frac{v_0}{\varrho} \left[\varrho^2 \nabla^2 K_n - \frac{\partial^2 K_n}{\partial 8^2} \right] \quad . \quad . \quad (54c)$$

$$\frac{\tau_\varrho}{2\eta} = \frac{v_0}{\varrho \sin \delta} \left[\frac{\partial^2 K_n}{\partial \delta \, \partial \lambda} - \operatorname{cotg} \delta \frac{\partial K_n}{\partial \lambda} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54d)$$

$$\frac{\tau_\delta}{2\eta} = \frac{1}{2 \sin \delta} \left[\frac{\partial v_0}{\partial \varrho} + \frac{w_0 - v_0}{\varrho} \right] \frac{\partial K_n}{\partial \lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54e)$$

$$\frac{\tau_\lambda}{2\eta} = \frac{1}{2} \left[\frac{\partial v_0}{\partial \varrho} + \frac{w_0 - v_0}{\varrho} \right[\frac{\partial K_n}{\partial \delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54f)$$

Lastly $43b$ combined with 53 gives

$$\frac{p_0}{\eta} = \varrho \frac{P}{\mu} + 2 \frac{\partial v_0}{\partial \varrho} + \varrho \frac{\partial^2 v_0}{\partial \varrho^2} - \frac{\partial w_0}{\partial \varrho} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

As we could expect, these formulae tend towards the formulae 30, 31. 32 and 33 of § 2 if n and ϱ become infinite and if for f the value given by 49 is introduced.

The equation 51, without the P terms, has already in 1935 been given by PEKERIS [2]) for the problem of convection in the Earth.

§ 5. *Short discussion of the solutions of §§ 2 and 4.*

Examining the results found for the problem of § 4 it strikes us that the equations for the elastic displacements or, in these case of viscous fluids, for the velocities, only depend on the order n of the spherical harmonic K_n and not on its coefficients. This, however, is no longer true for most of the stress-components; the fact that, with the exception of σ_ϱ, the differential quotients of K_n appear in the formulae makes them dependent on the coefficients and, therefore, on the particular shape of K_n. The value of σ_ϱ remains independent. This result makes it clear that even if the boundary conditions do not enforce a particular shape of K_n as e.g. in the case of the supposition of convection in the Earth because of the cooling at the surface without assuming a disturbance of the temperature over the surface, the different shapes K_n can adopt are not necessarily equivalent.

An analogous conclusion may be drawn for the problem of § 2. There also the results for the elastic displacements — resp. the velocities if our

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[2]) C. L. PEKERIS, Thermal convection in the interior of the Earth, Monthly Notices Royal Astronomical Soc., Geophys. Supplem., 3, 343—367 (1935).

problem has regard to a viscous fluid — are independent of the shape of the solution 14a of the equation 12 for K, but the stresses are not independent of this shape; except for σ_z the formulae for the stresses contain differential quotients of K which depend on the shape and on the coefficients of K.

In the formulae 47c and 54c for σ_λ we see the factor

$$\left[\varrho^2 \nabla^2 K_n - \frac{\partial^2 K_n}{\partial \delta^2}\right] \quad \text{(56a)}$$

By means of 38 this can of course also be written

$$\frac{1}{\sin^2\delta}\frac{\partial^2 K_n}{\partial \lambda^2} + \operatorname{cotg}\delta\,\frac{\partial K_n}{\partial \delta} \quad \text{(56b)}$$

or by means of 41

$$-n(n+1)K_n - \frac{\partial^2 K_n}{\partial \delta^2} \quad \text{(56c)}$$

In the shape of 56a it is simplest to see that this quantity is the second order differential quotient of K_n in E—W direction multiplied by ϱ^2, in the same way as the expression $\frac{\partial^2 K_n}{\partial \delta^2}$ represents the second order differential quotient of K_n in N—S direction multiplied by ϱ^2.

If F is a whole rational function of ϱ, the solution of the fundamental equation 44a for w_0 in § 4 is simple. For the term

$$F = S\varrho^{s-4}$$

we find in general

$$\left.\begin{aligned} w_0 = -\frac{S}{(n+1-s)(n-1-s)(n+s)(n+2+s)}\varrho^s + \\ + A\varrho^{n+1} + B\varrho^{n-1} + \frac{C}{\varrho^n} + \frac{D}{\varrho^{n+2}} \end{aligned}\right\} \quad \text{(57)}$$

in which A, B, C and D are integration-constants. In case this solution fails, the denominator of the coefficient of the first term being zero, i.e. for

$$\left.\begin{aligned} s &= n+1, \\ s &= n-1, \\ s &= -n, \\ \text{or}\quad s &= -(n+2) \end{aligned}\right\} \quad \text{(58a)}$$

we obtain

$$\left.\begin{aligned} w_0 = \frac{S}{(2s+1)[(n+1-s)(n+2+s)+(n-1-s)(n+s)]}\varrho^s \ln\varrho + \\ + A\varrho^{n+1} + B\varrho^{n-1} + \frac{C}{\varrho^n} + \frac{D}{\varrho^{n+2}} \end{aligned}\right\} \quad \text{(58b)}$$

The four other terms remain the same.

If F has the shape

$$F = S\,\varrho^{s-4}\,\ln\varrho$$

we get in general

$$\left.\begin{aligned} w_0 = & \frac{S}{(n+1-s)(n-1-s)(n+s)(n+s+2)}\,\varrho^s \ln\varrho - \\ & - \frac{(2s+1)\left[(n+1-s)(n+2+s)+(n-1-s)(n+s)\right]}{(n+1-s)^2(n-1-s)^2(n+s)^2(n+s+2)^2}\,S\varrho^s + A\varrho^{n+1} + \text{etc.} \end{aligned}\right\} \quad (59)$$

and for the cases of 58*a* we get terms with $\varrho^s \ln^2\varrho$, with $\varrho^s \ln\varrho$ and with ϱ^s combined with the usual A, B, C and D terms; their coefficients can be determined without difficulty.

The four integration-constants indicate that four conditions can be fulfilled in the boundary conditions, e.g. if the body is confined by two spheres, two quantities can be given for each of the two boundaries as σ_ϱ and τ_δ, τ_λ or V_ϱ and V_δ, V_λ; as τ_δ and τ_λ as well as V_δ and V_λ are connected by their being the components of the gradient of a potential function on the sphere, each pair can only fulfill one further condition. If the body is only limited by one sphere two conditions are necessary and sufficient for ensuring the continuity in the centre, and so two other conditions can be fulfilled at the surface.

If w_0 has been found the solution of e_0 from 45 is simple. If the right member of this equation is again a rational function of ϱ, a term having the shape

$$T\,\varrho^t$$

gives

$$e_0 = -\frac{m-2}{mt-2m+4}\,T\varrho^t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and for

$$T\,\varrho^t \ln\varrho$$

we obtain

$$e_0 = -\frac{m-2}{mt-2m+4}\,T\varrho^t\ln\varrho + \frac{(m-2)\,m}{(mt-2m+4)^2}\,T\varrho^t . \quad . \quad . \quad (61)$$

In solving (45) we should in general have to add a term

$$E\,\varrho^{\frac{2(m-2)}{m}}$$

in which E is an integration-constant, but as e_0 has not only to fulfill 45, which has been derived from 42 and 43*a*, but also the equation 43*b*, we have to substitute it likewise in this last equation. We then find a condition which can only be fulfilled if

$$n = \frac{2(m-2)}{m}$$

$$\text{or } n = -\frac{(3m-4)}{m}$$

$$m = 1.$$

The second of these conditions is impossible to fulfill as n has to be a positive integer, and the third is obviously also impossible. If we further do not admit a value of $m = \infty$ the fact that n has to be a positive integer leaves only the possibility

$$n = 1, \qquad m = 4. \qquad \text{(62a)}$$

The term to be added e_0 thus becomes

$$E\varrho. \qquad \text{(62b)}$$

This term indeed fulfills the three conditions 42, 43*a* and 43*b* and so our solution is valid. It obviously has one more integration-constant than normal, admitting one more boundary condition. This curious special case merits further investigation; it may be reserved for a future occasion.

Anatomy. — *On the frequent occurrence of double peaked index curves and the presence of allosexual indices in men and women.* By C. U. ARIËNS KAPPERS.

(Communicated at the meeting of November 24, 1945.)

Making curves of the l.br.indices of the head or skull of males it often struck us that these curves frequently show two principal peaks, even if the number of individuals is fairly large so that one might expect a normal curve. Registering the indices of females of the same races the same principal peaks may appear with this difference though, that, whereas in the male curve the peak of lower index value predominates, in the female curve the peak of higher value prevails (cf. Table I, fig. 13) or is the only one (cf. Table I, fig. 3). This induced us to examine more systematically the shape of the male and female index curves in various races.

Hitherto the data concerning the difference of the head or skull index of both sexes of the same races are chiefly confined to the statement that the average index of males usually is smaller than that of females.

In 42 groups of the 71 of which R. MARTIN mentions the average head index of both sexes, the male average index is smaller than the female average and the same holds good for 51 out of 71 average skull indices of both sexes. Moreover MARTIN doubts the reliability of those cases in which the average index of the females is recorded to be smaller [1]).

FRETS ('21 and '22) who examined a large number of Dutch men and women also found the average male index the smaller one. This author also called the attention to the fact that the spread of the index in the males is greater than in the females.

KLEIWEG DE ZWAAN ('42) recording the indices of both sexes from Bali and Lombok emphatically states that this people makes no exception on the general rule that the average male index is smaller than the female one.

In the following pages we shall analyse somewhat more exactly this general rule by means of frequency curves. Before doing so we should emphasize some points of importance in this matter.

For this purpose only such groups should be used, that may be considered as racially homogeneous (as far as this may be expected now a days) while the number of individuals should be large enough to give a trustworthy result. Another point is that the measurements on both sexes of a group are taken by the same person and thus with the same instrument.

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[1]) The only reliable example of such a case, according to our own experience, is found with the Armenians.

In data referring to skull measurements the sexing of the skulls should be reliable. Absolute reliability in this matter is obtained in post mortem material and with such skulls the origin of which is known by the character of the tombs.

Beginning with the cephalic indices (Table I) we call the attention to figure 1 referring to the indices of the Dutch boys and girls below 10 years and to fig. 4, giving the indices of their fathers and mothers. Both figures are constructed after the data recorded by FRETS ('21).

Fig. I shows that both sexes have the 80—81 [1]) peak in common, but that the average index value of the boys is slightly greater than that of the girls, on account of the 85 elevation in the former. In the adults of the same families this accessory peak of 85 no more exists. In both sexes of the adults two principal peaks occur of 78 and 80, but, whereas the peak of greater value (80) predominates in the female curve, the 78 peak is higher with the males.

Registering all the males (935) and females (1420) measured by FRETS (fig. 7) the male curve shows the 78 and 80 peak, generally found with Nordic groups, while the female curve only has the 80 peak. [2])

In the curves of English boys and girls made after the data of PEARSON and TIPPETT (fig. 2) this shifting of the male indices to the left is already visible below 10 years, the girls having both the 78 and 80 peaks, the boys the 78 peak only.

The shifting of the male index curve to the left is still more pronounced with the equally prevailing Nordic children of Massachusetts measured by WEST (fig. 3). Here the boys below 10 years have both a 77 (—78) and a 80 peak while the girls only show the 80 elevation also characteristic for the Dutch women in fig. 7.

The two different elevations characteristic of the Nordics are also evident in the index curves of the Faröer people measured by HANSEN (fig. 5). The 80 peak, the only one in the female curve, is indicated by a slight elevation in the male curve, in which the 78 peak prevails.

The male and female Norwegian curves (fig. 8) have the 80 peak in common while the male curve shows an additional 76 peak.

The male and female Persians from Isphahan, an Indo-Aryan group, measured by the KRISCHNERS, have their 75 and 78 peaks in common. but, whereas with the females the higher value (78) predominates, with the males the 75 peak prevails.

The far more dolichocephalic Papuans of the Bismarck archipelago. measured by CHINNERY (fig. 9), also give index curves in which both

[1]) In our curves the figure 70, 71 etc. stand for 70—70.9, 71—71.9 etc. The male indices are indicated by a continuous, the female indices by a broken line.

[2]) In the same fig. 7 we registered the males and females from the Zuiderzee island of Urk', recorded by PIEBENGA'. These indices are very different from those of the provinces of Holland, but also here the index of higher value (82) predominates with the females, the index of lower value (79) with the males.

sexes have the same elevations (72 and 75) while the lower index value (72) is far more outstanding with the males than with the females. The elevations with the Polynesians of the Marquesas (fig. 10) are interesting in so far as the lower average value of the male index is very evident here. The only index peak, which both sexes have in common is at 81. The other female indices have far higher values. MONTANDON's Ainu's (fig. 12) give a similar picture as SULLIVAN's Marquesans: both sexes have one peak (in casu 75) in common, but whereas the females have another elevation of higher value (77) the highest elevation of the males is at 73.

The picture of the Tonga islanders (fig. 11) is more instructive again since of the two principal peaks occurring in the female curve, the peak of lower value is the only one with the males. The most instructive picture, however, is given by VON EICKSTEDT's Santals (fig. 13). While the curves of the two sexes have the 75 and the 77 peaks in common the first peak predominates with the males, the second with the females.

KUBO's Koreans [3]) (fig. 14) and SCHOONHEYT's Batavian Javanese (fig. 15) show again the predominance of the lower values in the males who only share their higher indices with the females. DUNN's Hawaians of 15 years and older (fig. 16) also show an increase of the lower value indices (81—84) with the males.

With the Koryaks (fig. 17) both sexes show again two index peaks, the common peak of lower value (78) prevails with the males, the 81 peak with the females.

We finally give two Indian groups, the Seminoles (fig. 18) measured by KROGMAN [4]) and HALLOWELS Nas kapi, Indians from Labrador (fig. 19). Both groups again give double peak curves, the lower value prevailing with the males, the higher with the females. .

Proceeding to the curves referring to skulls we shall first deal with a section room series of the anatomical department of our University.

Dr. DE FROE examining this series of 196 male and 90 female skulls found the *average skull index* of both sexes about the same.

In connection with this statement it should be realized, however, that among the Dutch at least three different elements occur. Whereas the Nordic element is by far the greatest, in the South an Alpine and in the Eastern part of our country a Falic element occurs. The indices of these elements differ a good deal. Although Amsterdam has a prevailing Nordic population, there are a good number of other elements amongst them, the indices of which may influence the averages of the males and females.

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[3]) With KUBO's male Koreans it is questionable wether the rather numerous 78 index people is not a mixture at Koreans with Northern Chinese or Japanese.

[4]) This group is not entirely pure. 17 males and 23 females having some Sioux admixture.

Arranging the individual indices of DE FROE's material in index curves (Fig. 1 of Table II), it appears that the male and female curves have the 78 peak in common but that in addition to this the male curve has a considerable 76 peak, failing in the female curve. The cranial index of mesocephalics being about two points lower than the cephalic index the prevailing male skull indices in this series correspond exactly with the 78 and 80 cephalic index peaks of our male, the female cranial 78 peak with the cephalic 80 peak of our female Nordics (cf. Table I, fig. 7).

That notwithstanding the *average* of both sexes of this series is about the same may be explained by the assumption that among the male skulls in addition to the Nordic elements there is a larger number of Falic (82 index) and Alpine (84—87) elements. This clearly shows that in order to get reliable results one has to be sure about the homogeneity of the group and that frequency curves are far more instructive than averages.

As far as concerns homogeneity the Australian skulls measured by HRDLICKA (and others) offer a better material. So the index curves of the skulls from the state Victoria (Table II, fig. 2) show two peaks in both sexes (at 69 and 72) but in the male curve the lower value, in the female one the higher value again prevails. The same is seen in comparing all the Australian skulls of both sexes measured by this anthropologist (Table II, fig. 3), the female curve showing again its highest elevation at 72, the male one at 69. The Maori skulls, measured by various authors (Table II, fig. 4) show a prevailing number of dolichocephalics among the males, of mesocephalics among the females. In the curves of Teita Negro skulls of Miss KITSON (Table II, fig. 5), the 72 elevation of the females exceeds the same index of the males with whom the 70—71 index exceeds the corresponding female index.

Whereas the skull curves hitherto mentioned refer to races that are still existing so that the measurements may be obtained from well sexed section material, we shall consider also some extinct groups in which the sexing of the skulls may be subject to some doubt. While with ancient Egyptian material (Table II, fig. 6, 7 and 8) the sex may be well established with skulls originating from mummies, in those cases where the skulls were isolated no absolute certainty about its sex can be obtained, though other skeletal remains, especially the pelvis — if found with it — may be of great help.

In the small group of predystanic Badarian skulls measured by STOESSINGER (Table II, fig. 6) principally the same rule as with the above mentioned group is seen. The male and female curves have the peak of higher value (75) in common but the male curve has an even higher 71 peak, indicated in the female curve by a slight elevation only. Analogous relations are found with FAWCETT's and LEE's predynastic Nakada skulls (Table II, fig. 7) the female curve of which shows two elevations, one at 75 and one at 72. The peak of higher value fails in the male curve, where the lower values (between 71 and 73) are more numerous than

with the females. The curves of the eighteenth dynasty skulls (Table II, fig. 8) again show the dominance of the lower values with the males.

Of the extinct Guanches of the Canary islands two series of sexed skulls are at our disposal. In the larger series measured by HOOTON (Table II, fig. 9), the female indices culminate at 79 and 76, the male ones at 76 only. In VON BEHR's female skulls (Table II, fig. 10) the high value peak (78—79) corresponds with HOOTON's additional female peak. His male skulls show both peaks but here the lower value corresponds with HOOTON's single male peak.

The small series of Tasmanian skulls registered in Table II, fig. 11 shows an analogous relation as STOESSINGER's Badarians: the male skulls culminate in two peaks, the female ones in that of the higher value only.

Dealing with skull indices of both sexes we still may refer to the fact stated in one of our former papers ('43), that the ancient Viking skulls and the German "Reihengräber" and related Burgund skulls also give double peaked index curves. In these curves the 73 and 75 (sometimes 76) peaks dominate. In connection herewith it is interesting to note HAMY's [5]) statement that the average index of the female Merovingian and Carolingian skulls from Hardenthun and the Boulonnais is about 75, that of the male skulls 73.

Reviewing the results of our analysis of both sexes of various racial groups we do not only see that the average male index is lower than the female one but also that the male and female curves very often have two peaks in common and that the difference in these "associated peaks" is such that in the females the higher value, in the male the lower value peak is the most outstanding one (cf. Table I, fig. 13 and Table II, fig. 2 and 3).

This occurence of two not widely distant peaks in the curves of men and women probably is to be explained by the possibility, already mentioned by FRETS, that the males though usually inheriting the index of the father, may inherit also the index of the mother, while the females though usually inheriting the index of the mother may inherit the father's index, in other words by the *inheritance of allosexual indices*.

If the curve of one or each of the sexes has one peak only as usually occurs if larger numbers of people are measured, the higher value is characteristic for the female, the lower for the male (cf. Table I, fig. 3, 5, 7, 8, Table II, fig. 8, 9, 10, 11).

In connection with the different index of men and women it may be remembered that already AMMON ('99), FÜRST and RETZIUS ('02) referred to the *relation between body stature and head index* (*law of* AMMON), a point confirmed by PITTARD.

Considering this negative correlation between body size and skull index

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5) HAMY. Crânes Mérovingiens et Carolingiens. l'Anthropologie T. IV, 1893, p. 513.

(see also FAWCETT and PEARSON '98; CZEKANOWSKY '11, SCHUSTER '11, CRAIG '11 and MARTIN l.c.), we thought it worth while to examine also the data available for Pygmy races. Although in these groups also a difference in stature of the sexes exists, this difference usually is not so marked as it is in races of normal bodysize.

Cephalic curves relating to Pygmy races are given in Table III.

From these curves it appears that with the Aeta's of the Philippines (average stature of the males 143.6 cm, of the females 137.8; REED) the male index peak stands one point more to the left than the female index peak. With v. EICKSTEDT's Andamanese (average male stature 147.35 cm, female 138.5 cm) both curves show the same peaks but the peak of lower value is slightly higher in the male, the peak of higher value in the female. With the Semang of Malakka, examined by SHEBESTA, LEBZELTER, LAIDLAW, DUCKWORTH and MARTIN (average male stature 149 cm; female 140; SHEAT and BLAGDEN) the curves of both sexes have two equally high peaks (77 and 79) in common, but with the females the additional 82 peak is higher than with the males. A more pronounced difference in the same sense as observed with races of normal stature is found with the African Pygmies (average male stature 144 cm, female 136 cm) as appears from the curve of the male African Pygmies [6]) measured by SHEBESTA and JULIEN and LEBZELTER's female Basua and Bacwa Pygmies (Table III, fig. 4).

With BYLMER's Tapir as, who show a slightly greater difference in stature, the difference between the male and female curves is again more characteristic: the male indexcurve has two peaks, one at 77 and one at 79, the latter of which being the only with the females of this group [7]).

While the data mentioned above confirm the rule that the average index of the males usually is smaller than that of the females of the same race, the question remains if the difference in the male and female stature suffices to explain the fact or if the sexual factor in itself may also act a part.

For answering this question we shall compare the indices of shorter and taller people of the same sex.

The index curves of the shorter and taller male Islanders, examined by HANNESON and reproduced in textfigure 1 are not very convincing in this respect. The same may be said of the curves made of shorter and

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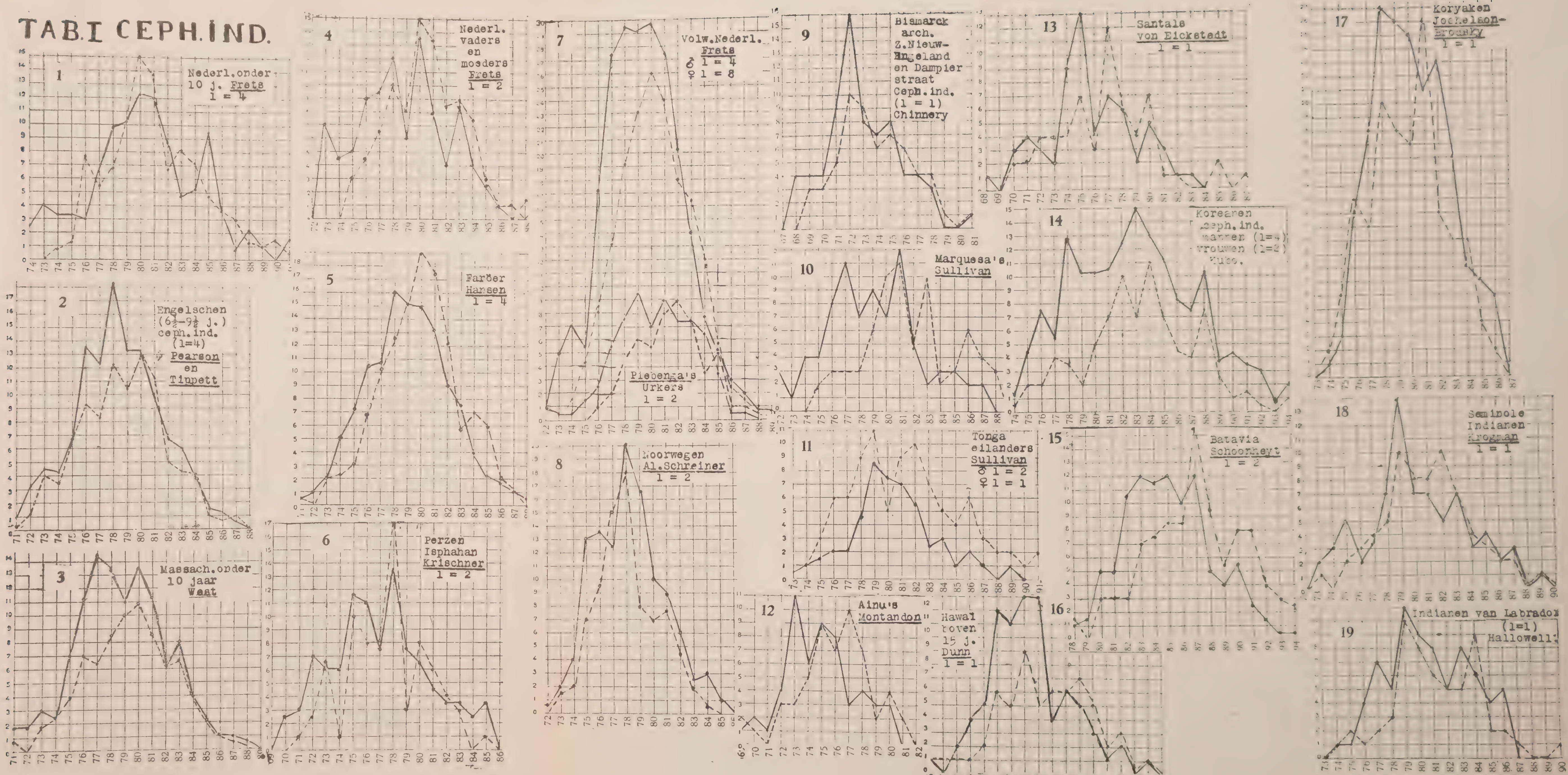
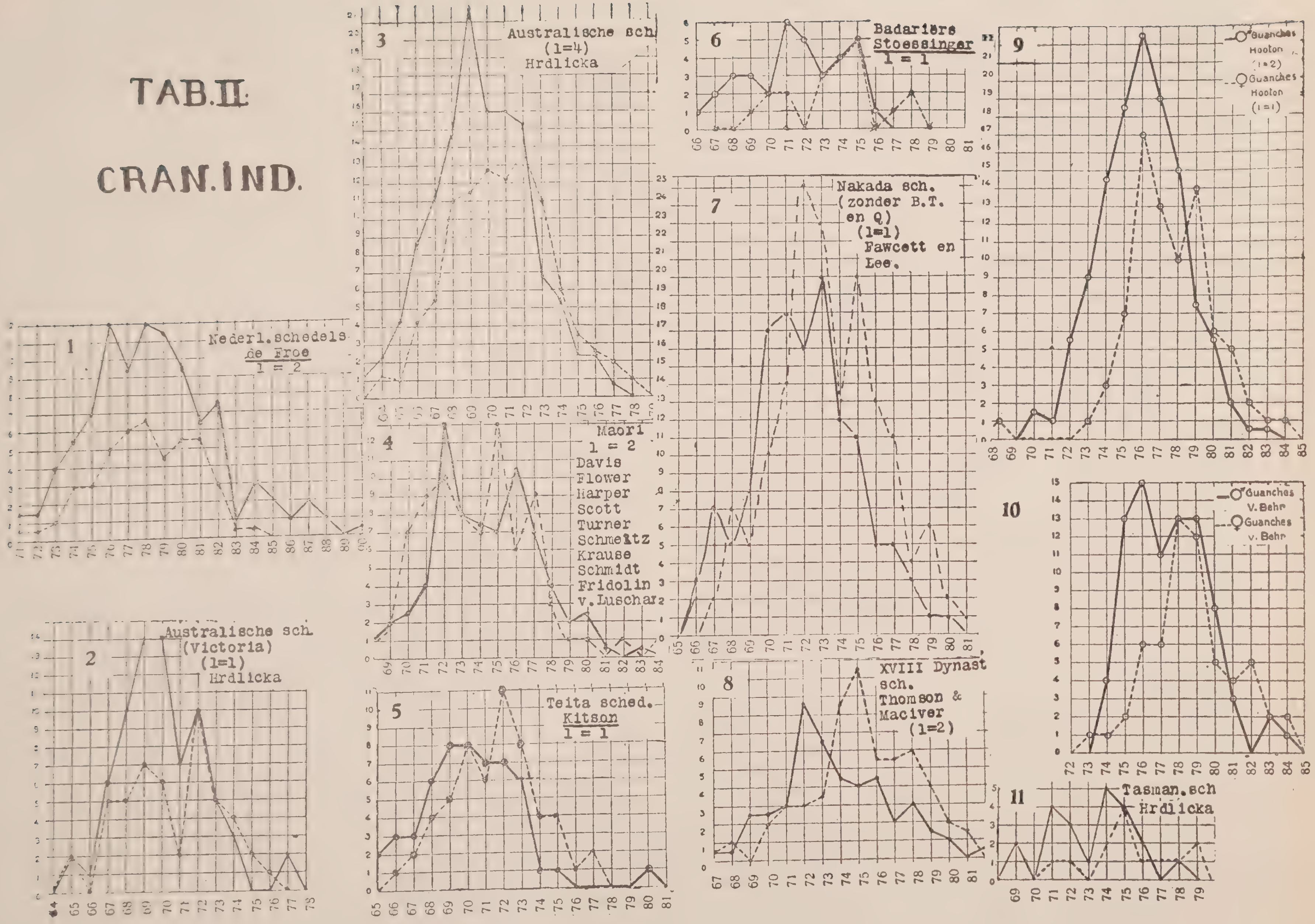
6) It is interesting to note that the female African Pygmies show exactly the same index peaks as the female Semang from Malakka: a strong argument in favour of the close kinship between these groups. The male Bacwa and Basua Pygmies of Africa only show the 77 peak, occurring also with the Semang (combined with a 79 peak).

7) Speaking of Pygmy races we will not omit to call the attention to BERGMANN's opinion concerning Pygmy races in the animal kingdom (cf. DE BEAUFORT, l.c. p. 43), viz. that the frequent occurrence of Pygmy races in the tropics may be due to the more favorable perspiration conditions for the relatively larger surface of small bodies.



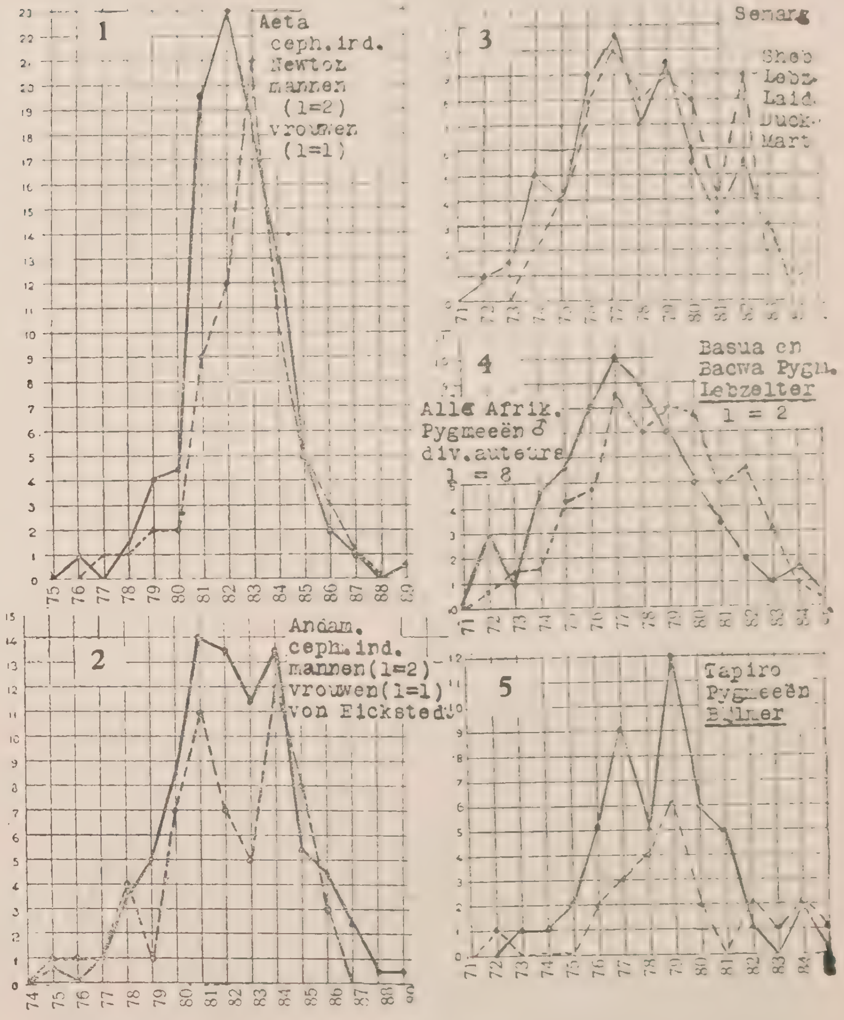
Male curves: ———; Female curves: — — — — —.

TAB. II: CRAN. IND.



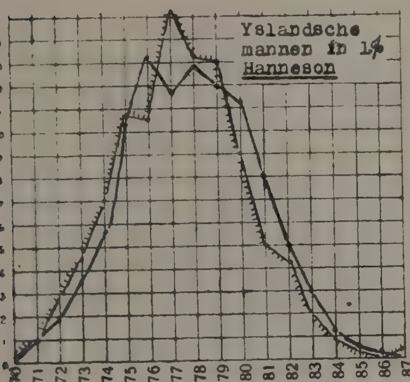
Male curves: ————; Female curves: — — — — —.

TAB. III CEPHAL. INDICES.

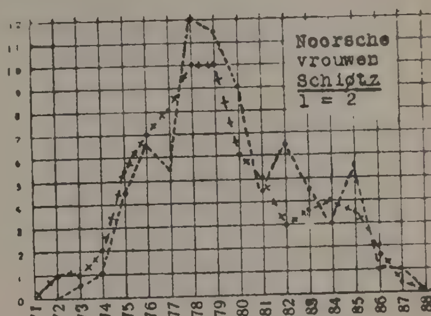


Male curves: ————; Female curves: — — — — —.

taller Norwegian women, the indices of which are recorded by SCHIOTZ (textfigure 2). With the latter we even find two additional elevations of higher index value (at 82 and 85) in the group of taller women. A similar absence of (negative) correlation between bodysize and index in women is also seen in comparing the female indices of Dutch girls (Table I, fig. 1) and adult Dutch women (Table I, fig. 4 and 7). Women generally are more inclined to keep the infantile index than men.

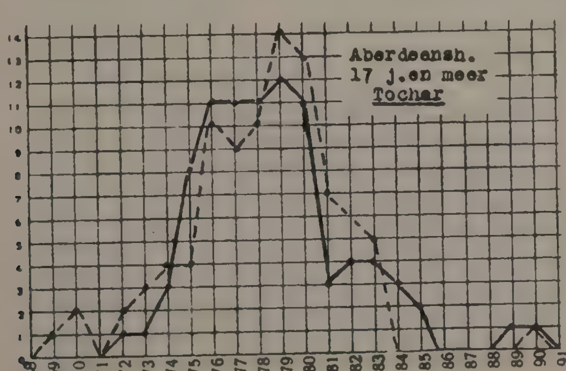


Textfig. 1.
Cephalic index curves of male Islanders after HANNESON. Continuous line: indices of men of 156—172 cm bodylength. Combed line: indices of men of 172—193 cm bodylength.



Textfig. 2.
Cephalic index curves of Norvegian women after the data of SCHIøTZ. Dotted line: indices of women of 157—162,8 cm bodylength. Crossed line: indices of women of 162,8—176,4 cm bodylength.

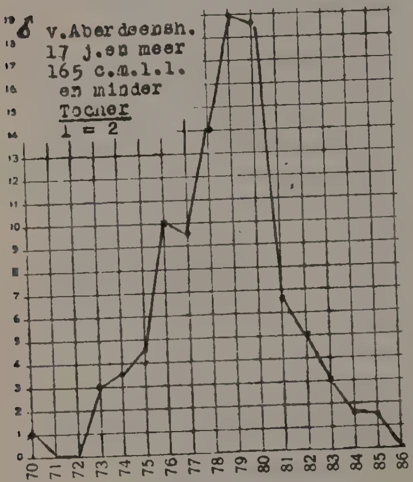
For examining the correlation between body stature and index in the same race, TOCHER's data referring partly to inhabitants of Scotch asylums, partly to normal people of the same towns are of great importance.



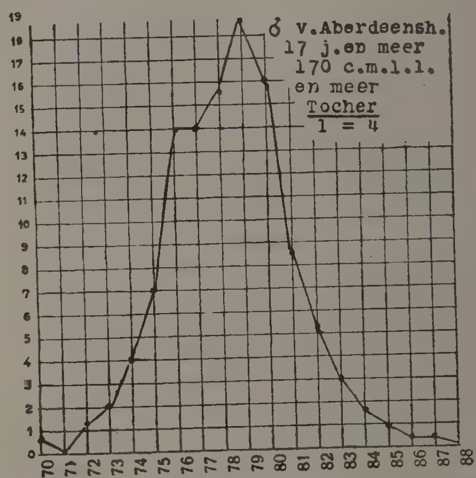
Textfig. 3.
Cephalic indices of normal adults from Aberdeenshire after TOCHER's measurements. Continuous line: male indices; broken line: female indices.

We shall deal first with the two sexes of normal people from Aberdeenshire. Of the 88 women of 17 years and older measuring from 1704 to 1415 mm, the majority measured less than 1650 mm. For a comparison with these women we took an equal number of normal men of the same age from the same places. The bodysize of these men, taken at random, varied from 1812 to 1497 mm, the majority being more than 1650 mm.

The index curves reproduced in textfigure 3 show two peaks, one at 76, the other at 79. The male curve differs from the female by a lower peak of high value and a slightly higher peak of low value, as usually found in comparing both sexes of the same normal group of people. In order to know if this relation may be explained by different bodysize



Textfig. 4.
Cephalic indices of adult normal men of 165 cm and less from Aberdeenshire. After TOCHER's measurements.



Textfig. 5.
Cephalic indices of adult men from Aberdeenshire of 170 cm and more. After TOCHER's measurements.

only we made curves of all normal men from Aberdeenshire of 17 years and older whose bodysize was 165 cm or less and of all normal men from this county of the same age with a bodysize of 170 cm and more.

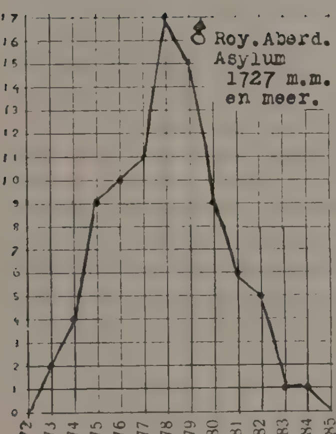
Textfigure 4 and 5 show that in both curves two elevations occur, one at 76, the other at 79, and that the lower value index (76) is far more numerous in the taller men than in the shorter ones, thus confirming the negative correlation between stature and head index.

In order to verify the correlation between index and stature we made also curves of the inhabitants of the Aberdeenshire asylum, although this material is not so trustworthy for our purpose, in the first place since it refers to pathological individuals and on account of the fact that the ages of this group are not mentioned by TOCHER.

As with the Norwegian and Dutch women of different stature and age the difference in the index of the shorter and taller women is small [8]).

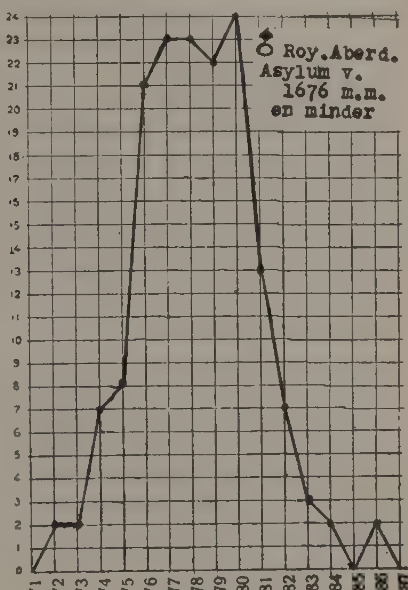
With the male asylum inhabitants the correlation between the stature and index is more pronounced (cf. textfigures 6 and 7).

In the curve of males of 1676 mm and shorter (fig. 7) two elevations occur, one at 77—78 and one at 80 (the cephalic peaks characteristic of the Nordic race). In the curve of taller men (fig. 6) the 78 peak is the



Textfig. 6.

Cephalic indices of male inhabitants of the Royal Aberdeenshire Asylum of 1727 mm body length and taller.



Textfig. 7.

Cephalic indices of male inhabitants of the Royal Aberdeenshire Asylum of 1676 mm body length and shorter.

only one. The disappearance of the peak of higher value (80) in figure 6 is in harmony with the greater bodysize of this group.

This confirms that bodysize acts an important part in the shape of the index curves of males of the same race. The double peaked curves in Table I and II, however, show that both male and female individuals may inherit allosexual indices, although the majority of the males inherits the male, the majority of the females the female index, as is also stated by FRETS ('19).

This conclusion is not minimized by the fact that the double peaked

[8]) The fact that bodysize (and age) are less influential in women than in men confirms BOLK's opinion that women keep the infantile features longer than men.

shape of index curves usually changes into a single peaked one if larger numbers of individuals are registered. Besides in such cases the female peak always is of higher value than the male peak [9]) and the curves usually acquire the character of skew curves.

Fully agreeing with HANNESON (l.c. p. 150) where he says "man muss HALFDAN BRYN Recht geben wenn er sagt dass die Form der Kurve nicht auf Zufall beruhen kann", we want to state especially that it is wrong to reduce an index curve to a normal curve as frequently done by anthropologists.

Summary.

The presence of two principal peaks in the l/br. index curves may be frequently explained by the occurrence of *allosexual indices* in both sexes. In the female curves the peak of larger indices, in the male curves the peak of smaller indices prevails. This phenomenon may be correlated with sexual differences in stature.

Résumé.

La présence de deux sommets principaux dans les courbes des indices horizontaux s'explique en plusieurs cas par la présence d'*indices allosexuels* dans les deux sexes. Dans les courbes femelles le sommet prédominant est celui des grands indices, dans les courbes des mâles celui des indices moins élevés. Ce phénomène peut correspondre avec des différences sexuelles dans la stature.

Zusammenfassung.

Die Anwesenheit zweier Hauptgipfel in L/Br Indexkurven ist in manchen Fällen zu erklären durch das Vorkommen *allosexueller Indizes* in beiden Geschlechtern, wobei in der weiblichen Kurve der Gipfel der grösseren, in der männlichen Kurve der Gipfel der kleineren Indizes überwiegt. Diese Erscheinung kann korreliert sein mit einem sexuellen Unterschied in der Statur.

Samenvatting.

De aanwezigheid van twee hoofdtoppen in l/br. index curven is in vele gevallen te verklaren door het voorkomen van *allosexûeele indices* in beide geslachten, waarby in de vrouwelijke curve de top der grootere indices, in de mannelijke curve de top der kleinere indices overweegt. Dit kan gecorreleerd zijn met de statuurverschillen der beide geslachten.

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9) The only exception on this rule according to our personal experience is found with the Armenians.

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Mathematics. — *Over simplices, die de ligging van* SCHLÄFLI *bezitten.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 27, 1945.)

1. Reeds PONCELET heeft een driedimensionaal analogon gegeven van de stelling van DESARGUES: als de verbindingslijnen van toegevoegde hoekpunten van twee tetraëders door één punt gaan, dan liggen de snijlijnen van overeenkomstige zijvlakken in één vlak. Later is gebleken, dat de figuur van twee perspectieve tetraëders een te speciale generalisatie is van de figuur van DESARGUES: terwijl twee perspectieve driehoeken steeds elkaars polaire figuur zijn ten opzichte van een kegelsnede en omgekeerd, zijn twee tetraëders, welke polair toegevoegd zijn ten opzichte van een quadratisch oppervlak in het algemeen niet perspectief; wel vormen de verbindingslijnen van toegevoegde hoekpunten een bijzonder lijnenquadrupel: zij zijn volgens een van CHASLES afkomstige stelling *hyperboloïdisch*. Naar het schijnt is het eerst door HERMES het theorema uitgesproken, dat de generalisatie betekent van dat van DESARGUES: als de verbindingslijnen van toegevoegde hoekpunten van twee viervlakken hyperboloïdische ligging hebben, dan geldt hetzelfde voor de snijlijnen van overeenkomstige vlakken.

SCHLÄFLI heeft de stelling van CHASLES uitgebreid tot R_n; de generalisatie van vier rechten in hyperboloïdische ligging is daarbij de figuur van $n+1$ rechten, die de eigenschap hebben, dat elk der (∞^{n-2}) R_{n-2}, welke n der rechten snijdt, ook met de overige rechte een punt gemeen heeft. Een analytisch bewijs van de stelling van SCHLÄFLI is gegeven door BERZOLARI. In navolging van hem zegt men van $n+1$ rechten met de genoemde eigenschap, dat zij *de ligging van* SCHLÄFLI bezitten. Analytisch kan men de eigenschap uitdrukken door de voorwaarde, dat de matrix van de GRASSMANNse coördinaten der $n+1$ rechten *de rang* n bezit. Dualistisch definieert men de ligging van SCHLÄFLI voor $n+1$ ruimten R_{n-2}. De stelling van BERZOLARI luidt dan: bezitten de verbindingslijnen van overeenkomstige hoekpunten van twee simplices in R_n de ligging van SCHLÄFLI, dan geldt hetzelfde voor de doorsneden van overeenkomstige zijruimten. De simplices zijn dan ook elkaars poolfiguur ten opzichte van een quadratische variëteit en uit één der drie genoemde eigenschappen volgt de juistheid der beide overige [1]).

[1]) SCHLÄFLI, Erweiterung des Satzes, dass zwei polare Dreiecke perspectivisch liegen, auf eine beliebige Zahl von Dimensionen. Journ. f. Mathem. **65**, 189 (1866).

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Id. Über Simplexe in SCHLÄFLIscher Lage, Bull. Soc. Sc. de Cluj, **5**, 488—491 (1931).

2. Wij beperken ons nu voorlopig tot R_3. Hebben de verbindingslijnen $P_i Q_i$ $(i = 1, 2, 3, 4)$ van twee viervlakken $P_1 P_2 P_3 P_4$ en $Q_1 Q_2 Q_3 Q_4$ de ligging van SCHLÄFLI, dan geldt dus het zelfde voor de snijlijnen van overeenkomstige zijvlakken $P_i P_j P_k$ en $Q_i Q_j Q_k$. Wij vragen ons af of men *een vergelijkbare eigenschap kan aangeven voor de zes paren overeenkomstige ribben* $l_{ij} \equiv P_i P_j$ *en* $m_{ij} \equiv Q_i Q_j$.

Door de lijncoördinaten van l_{ij} en m_{ij} te interpreteren als puntcoördinaten in R_5 worden deze ribben afgebeeld op de hoekpunten van twee simplices L_{ij} en M_{ij} in R_5. Wij zullen aantonen: *de simplices L en M hebben de ligging van* SCHLÄFLI.

Voor het bewijs kiezen wij P_1 P_2 P_3 P_4 tot fundamentaaltetraëder. De coördinaten van Q_i zijn $(a_i\, b_i\, c_i\, d_i)$ [2]). Duiden wij de Plückerse lijncoördinaten $p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}$ kortheidshalve resp. aan met $p_1, p_2, p_3, p_4, p_5, p_6$ (en analoog daarmee de punten $L_{12}, L_{13} \ldots, M_{12}, M_{13} \ldots$ met $L_1, L_2 \ldots, M_1, M_2 \ldots$), dan is de matrix van de coördinaten der vier verbindingslijnen $P_i Q_i$ als volgt

$$\begin{array}{c|cccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ P_1 Q_1 & b_1 & c_1 & d_1 & 0 & 0 & 0 \\ P_2 Q_2 & -a_2 & 0 & 0 & 0 & -d_2 & c_2 \\ P_3 Q_3 & 0 & -a_3 & 0 & d_3 & 0 & -b_3 \\ P_4 Q_4 & 0 & 0 & -a_4 & -c_4 & b_4 & 0 \end{array} \quad \ldots \quad (1)$$

Nodig en voldoende voor de hyperboloïdische ligging van de vier rechten is dat deze matrix de rang *drie* heeft, dus dat b.v. de coördinaten van $P_4 Q_4$ lineair afhankelijk zijn van die der drie overige rechten. Stelt men

$$P_4 Q_4 = \lambda_1 \ P_1 Q_1 + \lambda_2 P_2 Q_2 + \lambda_3 P_3 Q_3,$$

dan is dus

$$\lambda_1 : \lambda_2 = a_2 : b_1 \ , \ \lambda_1 : \lambda_3 = a_3 : c_1 \ , \ \lambda_1 : \lambda_4 = a_4 : d_1 \quad \ldots \quad (2)$$

zodat de drie voorwaarden:

$$\lambda_3 d_3 = \lambda_4 c_4 \ , \ \lambda_2 d_2 = \lambda_4 b_4 \ , \ \lambda_2 c_2 = \lambda_3 b_3 \quad \ldots \quad (3)$$

vervuld moeten zijn, waaruit na eliminatie van λ_i de volgende onafhankelijke betrekkingen ontstaan:

$$a_3 c_4 d_1 = a_4 c_1 d_3 \ , \ a_4 b_1 d_2 = a_2 b_4 d_1 \ , \ a_2 b_3 c_1 = a_3 b_1 c_2. \quad . \quad (4)$$

Uiteraard gelden ook alle door cyclische verwisseling hieruit te ver-

[2]) Wij veronderstellen in het volgende, dat de beide viervlakken naar de terminologie van WEISS (Math. Zeischr. **33**, 389) *vrij* van elkaar zijn gelegen, d.w.z. dat geen hoekpunt van één der viervlakken gelegen is in een zijvlak van het andere, dat niet ligt tegenover het toegevoegde hoekpunt. Analytisch komt dat hierop neer, dat in de matrix $(a\ b\ c\ d)$ de elementen buiten de hoofddiagonaal en ook de minoren van die elementen ongelijk aan nul zijn.

krijgen relaties; zij zijn echter afhankelijk van de drie betrekkingen (4). Zo ontstaat b.v. na vermenigvuldiging van de overeenkomstige leden (daar $a_2\, a_3\, a_4\, b_1\, c_1\, d_1 \neq 0$ is):

$$b_3\, c_4\, d_2 = b_4\, c_2\, d_3. \qquad (5)$$

Wij richten nu onze aandacht op de simplices L en M in R_5. Het punt L_1 heeft de coördinaten $p_1 = 1$, $p_2 = p_3 = p_4 = p_5 = p_6 = 0$; die van het punt M_1 zijn de minoren uit de matrix

$$\left\| \begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{matrix} \right\|.$$

De verbindingslijn $L_1 M_1$ heeft 15 Grassmannse lijncoördinaten q_{12} waarvan er blijkbaar slechts 5 ongelijk aan nul zijn, n.l. $q_{1i}\,(i = 2, 3, 4, 5, 6)$; daarvoor geldt

$$\left.\begin{aligned} q_{12} &= p_2^1 = p_{13}^1 = a_1\, c_2 - a_2\, c_1 \\ q_{13} &= p_3^1 = p_{14}^1 = a_1\, d_2 - a_2\, d_1\,, \text{ etc.} \end{aligned}\right\} \qquad (6)$$

waarbij de bovenste index het nummer der verbindingslijn aangeeft. De matrix der coördinaten van de zes verbindingslijnen $L_i M_i$ ziet er dan als volgt uit

$$\begin{array}{c} \\ L_1 M_1 \\ L_2 M_2 \\ L_3 M_3 \\ L_4 M_4 \\ L_5 M_5 \\ L_6 M_6 \end{array} \begin{array}{ccccccccccccccc} q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & q_{23} & q_{24} & q_{25} & q_{26} & q_{34} & q_{35} & q_{36} & q_{45} & q_{46} & q_{56} \\ p_{13}^1 & p_{14}^1 & p_{34}^1 & p_{42}^1 & p_{23}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_{12}^2 & 0 & 0 & 0 & 0 & p_{14}^2 & p_{34}^2 & p_{42}^2 & p_{23}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p_{12}^3 & 0 & 0 & 0 & -p_{13}^3 & 0 & 0 & 0 & p_{34}^3 & p_{42}^3 & p_{23}^3 & 0 & 0 & 0 \\ 0 & 0 & p_{12}^4 & 0 & 0 & 0 & -p_{13}^4 & 0 & 0 & p_{14}^4 & 0 & 0 & p_{42}^4 & p_{23}^4 & 0 \\ 0 & 0 & 0 & -p_{12}^5 & 0 & 0 & 0 & -p_{13}^5 & 0 & 0 & -p_{14}^5 & 0 & -p_{34}^5 & 0 & p_{23}^5 \\ 0 & 0 & 0 & 0 & -p_{12}^6 & 0 & 0 & 0 & -p_{13}^6 & 0 & 0 & -p_{14}^6 & 0 & -p_{34}^6 & -p_{42}^6 \end{array} \qquad (7)$$

Zij heeft de rang *vijf*, als er coëfficiënten $\lambda_i\,(i = 1, 2, 3, 4, 5, 6)$ bestaan die niet alle nul zijn en waarvoor geldt: $\Sigma\, \lambda_i\, L_i\, M_i = 0$.

Hieruit volgt ten eerste

$$\left.\begin{aligned} \lambda_1 : \lambda_2 = p_{12}^2 : p_{13}^1\ ;\ \lambda_1 : \lambda_3 = p_{12}^3 : p_{14}^1\ ;\ \lambda_1 : \lambda_4 = p_{12}^4 : p_{34}^1\ ; \\ \lambda_1 : \lambda_5 = p_{12}^5 : p_{42}^1\ ;\ \lambda_1 : \lambda_6 = p_{12}^6 : p_{23}^1 \end{aligned}\right\} \qquad (8)$$

De verhoudingen der λ zijn hierdoor bepaald; de coördinaten p_{ij}^k moeten dus nog voldoen aan *tien* onafhankelijke relaties. De eerste hiervan luidt

$$p_{14}^2\, \lambda_2 = p_{13}^3\, \lambda_3$$

of wel

$$p_{13}^1\, p_{14}^2\, p_{12}^3 = p_{14}^1\, p_{12}^2\, p_{13}^3. \qquad (9)$$

De overige kunnen uit de matrix (7) of door cyclische verandering uit (9) worden afgeleid.

Om onze stelling te bewijzen hebben wij nu slechts aan te tonen, dat de betrekking (9) en de analoge een gevolg zijn van de relaties (4). Wij kunnen ons daarvoor tot (9) zelf beperken. Uit (6) volgt

$$\left.\begin{array}{ll} p^1_{13} = a_1 c_2 - a_2 c_1 & p^1_{14} = a_1 d_2 - a_2 d_1 \\ p^2_{14} = a_1 d_3 - a_3 d_1 & p^2_{12} = a_1 b_3 - a_3 b_1 \\ p^3_{12} = a_1 b_4 - a_4 b_1 & p^3_{13} = a_1 c_4 - a_4 c_1 \end{array}\right\} \quad . \; . \; . \quad (10)$$

Uit (4) volgt

$$a_2 c_1 = a_3 c_2 \frac{b_1}{b_3}, \; a_3 d_1 = a_4 d_3 \frac{c_1}{c_4}, \; a_4 b_1 = a_2 b_4 \frac{d_3}{d_1} \quad . \; . \quad (11)$$

zodat men met behulp van (5) heeft:

$$p^1_{13} p^2_{14} p^3_{12} = (a_1 c_2 - a_2 c_1)(a_1 d_3 - a_3 d_1)(a_1 b_4 - a_4 b_1) =$$

$$= \frac{b_4 c_2 d_3}{b_3 c_4 d_2} (a_1 b_3 - a_3 b_1)(a_1 c_4 - a_4 c_1)(a_1 d_2 - a_2 d_1)$$

$$= (a_1 b_3 - a_3 b_1)(a_1 c_4 - a_4 c_1)(a_1 d_2 - a_2 d_1) = p^1_{14} p^2_{12} p^3_{13}$$

waarmee (9) bewezen is.

3. De aangetoonde ligging van SCHLÄFLI voor de simplices L en M in R_5 kan als volgt worden geïnterpreteerd. De verbindingslijnen $L_i M_i$ hebben de eigenschap, dat elke R_3 die vijf der lijnen snijdt, ook de zesde ontmoet. In de stralenruimte van R_3 betekent dat dus het volgende:

Als de tetraëders $P_1 P_2 P_3 P_4$ en $Q_1 Q_2 Q_3 Q_4$ de ligging van SCHLÄFLI *bezitten, dan geldt: de zes bundels lineaire complexen, die door twee toegevoegde ribben $P_i P_j$ en $Q_i Q_j$ bepaald zijn, hebben de eigenschap, dat elke drieledige lineaire verzameling van lineaire complexen, welke met vijf der bundels een complex gemeen heeft, ook een complex van de zesde bundel bevat.* Men kan in R_5 ook de figuur beschouwen, die aan de simplices L en M wordt toegevoegd door de poolcorrelatie t.o.v. de quadratische variëteit Ω met de vergelijking $p_1 p_4 + p_2 p_5 + p_3 p_6 = 0$.

Aan elk der verbindingslijnen $L_i M_i$ is een R_3 toegevoegd. Deze zes R_3's hebben de eigenschap, dat elke rechte, die vijf dezer ruimten snijdt, ook met de zesde een punt gemeen heeft. De interpretatie in de lijnenruimte luidt dan: *twee overeenkomstige ribben $P_i P_j$ en $Q_i Q_j$ van de beide tetraëders $P_1 P_2 P_3 P_4$ en $Q_1 Q_2 Q_3 Q_4$ bepalen een lineaire driedimensionale verzameling van lineaire complexen, n.l. die verzameling, waarvan de transversalen van $P_i P_j$ en $Q_i Q_j$ de dragers der speciale complexen zijn. Hebben de tetraëders P en Q de ligging van* SCHLÄFLI, *dan geldt: elke bundel lineaire complexen, die met vijf der verzamelingen een complex gemeen heeft, bevat ook een complex van de zesde verzameling.*

Een bijzonder geval van deze eigenschap ontstaat als men haar toepast op een bundel van speciale complexen, dus als men in R_5 een transversaal

der zes R_3's kiest, welke op Ω is gelegen. Blijkbaar is er een eindig aantal van deze transversalen. Wij krijgen dan de volgende eenvoudige uitspraak: *hebben de tetraëders* $P_1\, P_2\, P_3\, P_4$ *en* $Q_1\, Q_2\, Q_3\, Q_4$ *de ligging van* SCHLÄFLI, *dan heeft elke waaier, welke van vijf der lineaire congruenties* $(P_i P_j,\; Q_i Q_j)$ *een rechte bevat, ook met de zesde congruentie een rechte gemeen.*

Hebben de simplices L en M de ligging van SCHLÄFLI, dan bestaat er een quadratische variëteit V in R_5 zodanig, dat L en M reciproke poolfiguren zijn t.o.v. V; dat wil dus zeggen, dat de poolruimte van L_i (resp. M_i) t.o.v. V samenvalt met de overstaande zijruimte van M_i (resp. L_i). Nu is het lineair complex, dat vijf ribben van het tetraëder P bevat, n.l. alle ribben behalve $P_i P_j$, niets anders dan het speciale complex met $P_k P_l$ tot drager. Wij vinden dus: *hebben de viervlakken* P *en* Q *de ligging van* SCHLÄFLI, *dan bestaat er een quadratisch complex* V *met de eigenschap, dat de ribben* $P_i P_j$ *(resp.* $Q_i Q_j$*) tot lineair poolcomplex t.o.v.* V *bezitten de speciale complexen met* $Q_k Q_l$ *(resp.* $P_k P_l$*) tot dragers.*

4. De stelling kan worden uitgebreid tot en op analoge wijze bewezen voor ruimten van meer dimensies. Beschouwt men b.v. twee simplices P en Q in R_4, dan kan men hun ribben (of hun zijvlakken) afbeelden op de hoekpunten van twee simplices L en M in R_9; hebben P en Q de ligging van SCHLÄFLI, dan geldt hetzelfde voor L en M.

5. Om de ligging van SCHLÄFLI voor twee *vrij* gelegen simplices P en Q in R_n analytisch vast te leggen, kan men steeds P tot coördinatensimplex kiezen, waarna de hoekpunten van Q de coördinaten $(a_i\, b_i\, c_i \ldots\ldots)$ verkrijgen. De lijncoördinaten van de verbindingslijnen $P_i Q_i$ kunnen dan worden samengevat in een matrix van $n+1$ rijen en $\frac{1}{2} n\,(n+1)$ kolommen, die de gedaante (1) heeft. De voorwaarden, waaronder deze matrix de rang n bezit, kunnen op dezelfde wijze worden afgeleid als boven geschiedde voor $n=3$ en $n=5$, n.l. door de eis te stellen, dat er tussen de $n+1$ rijen een lineaire afhankelijkheid bestaat. Uit n der betrokken vergelijkingen kunnen de verhoudingen der coëfficiënten λ_i worden bepaald en er ontstaan dan $\frac{1}{2} n\,(n+1) - n = \frac{1}{2} n\,(n-1)$ onafhankelijke relaties tussen de getallen a_i, b_i, ... om de ligging van SCHLÄFLI analytisch te omschrijven. Deze relaties hebben ten duidelijkste alle de vorm, welke wij voor $n=3$ en $n=5$ gevonden hebben (4; 9) n.l.

$$a_2\, b_3\, c_1 = a_3\, b_1\, c_2 \quad \ldots\ldots\ldots \quad (12)$$

Bij algemene keuze van het coördinatenstelsel wordt a_2 niets anders dan de bilineaire invariant $(Q_2 p_1)$ van het punt Q_2 en het tegenover P_1 gelegen zijvlak p_1 van het tetraëder P, welke invariant ongelijk nul is, omdat P en Q vrij liggen. De relatie (12) gaat dan over in

$$(Q_2\, p_1)\,(Q_3\, p_2)\,(Q_1\, p_3) - (Q_3\, p_1)\,(Q_1\, p_2)\,(Q_2\, p_3) = 0. \quad \ldots \quad (13)$$

Het linkerlid hiervan is een uitdrukking, zoals die door STUDY is ingevoerd en waaraan WEISS de naam *Dreierring* heeft gegeven [3]). In het bovenstaande is dus opgesloten een eenvoudig bewijs van de door WEISS afgeleide stelling, welke uitspreekt, dat de ligging van SCHLÄFLI kan worden weergegeven door relaties van de vorm (13), dus door relaties zoals zij voor $n = 2$, in de figuur van DESARGUES, voorkomen.

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[3]) STUDY, Einleitung in die Theorie der Invarianten (1923), 62; WEISS, Math. Zeitschr. **33**, 389—391.

Mathematics. — *Een involutie in de lijnenruimte.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 27, 1945.)

§ 1. Gegeven zijn twee quadratische regelscharen R_1 en R_2, respectievelijk gelegen op de hyperboloïden H_1 en H_2. Een rechte l der ruimte snijdt twee beschrijvenden van R_1 en twee van R_2; deze vier rechten hebben behalve l nog een tweede transversaal l'. De involutorische verwantschap tussen l en l' is onderzocht door JAN DE VRIES, door SCHAAKE en door BONE [1]), die haar voornaamste eigenschappen hebben afgeleid. Wij komen hier op deze verwantschap terug ten eerste, omdat de genoemde auteurs enige voor de hand liggende vragen onbesproken hebben gelaten (met name die naar de aard der figuur, die toegevoegd is aan een willekeurig regeloppervlak en aan een willekeurig complex) en vervolgens om aan te tonen, dat naast de meetkundige behandeling ook een meer analytische bespreking mogelijk is, die onder meer in staat stelt de meetkundige plaats der dubbelrechten uitvoeriger te onderzoeken. In § 2 geven wij een samenvatting van de door de drie schrijvers verkregen resultaten.

§ 2. Door SCHAAKE zijn voor een involutie in de lijnenruimte drie karakteristieke getallen gedefinieerd en dezelfde heeft die involuties bepaald, waarvoor deze getallen nul of één zijn. Het blijkt, dat de onderhavige involutie diegene is, waarvoor de karakteristieke getallen alle gelijk aan één zijn [2]).

Singuliere rechten van de involutie zijn ten eerste de stralen der op H_1 resp. H_2 gelegen, met R_1 resp. R_2 verbonden regelscharen S_1 en S_2; aan elk dezer rechten is een lineaire congruentie toegevoegd. Voorts is er een congruentie (4,4) van singuliere stralen, aan elk waarvan een waaier is toegevoegd; deze congruentie is opgebouwd uit waaiers, waarvan de top een gemeenschappelijk punt van H_1 en H_2 en het vlak een gemeenschappelijk raakvlak van deze oppervlakken is.

De dubbelrechten der involutie vormen een quadratisch complex.

Aan een waaier is een cubisch regelvlak, aan een schoof een congruentie (2,3) en aan een veld een (3,2) toegevoegd.

Beeldt men de rechten der ruimte af op de punten van een quadratische

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[1]) JAN DE VRIES, Quadratische involuties in de stralenruimte, Versl. Kon. Akad. v. Wetensch., Amsterdam, **27**, 842—848 (1918); SCHAAKE, De ranggetallen van een straleninvolutie in de ruimte, Versl. Kon. Akad. v. Wetensch., Amsterdam, **33**, 804—815 (1924); BONE, Wiskundige Opgaven **13**, 150—153 (1925).

[2]) Voor involuties, waarvan één of twee karakteristieke getallen *twee* zijn, zie VAN DIJK, Involuties in de stralenruimte, diss. Groningen (1935).

vierdimensionale variëteit Ω in R_5, waardoor aan l resp. l' de punten P en P' zijn toegevoegd, dan is de verbindingslijn PP' een transversaal over twee, de involutie bepalende, vaste vlakken V_1 en V_2. De doorsneden van V_1 en V_2 met Ω corresponderen daarbij met de regelscharen S_1 èn S_2.

§ 3. Het is aan de laatst genoemde eigenschap, dat wij de analytische behandeling der involutie willen verbinden. Voert men de lijncoördinaten p_{ij} van PLÜCKER in, dan luidt de vergelijking van Ω

$$p_{12}\,p_{34} + p_{13}\,p_{42} + p_{14}\,p_{23} = 0.$$

Wij zullen nu coördinaten x_i $(i = 1,, \ldots\ldots 6)$ in R_5 invoeren, die lineair afhangen van p_{ij} en wel zó, dat de vlakken V_1 en V_2 resp. door de vergelijkingen

$$x_4 = x_5 = x_6 = 0 \text{ en } x_1 = x_2 = x_3 = 0 \,. \quad . \quad . \quad . \quad . \quad (1)$$

worden voorgesteld. De variëteit Ω krijgt dan een vergelijking van algemene gedaante

$$Q \equiv \sum_{i,j}^{6} a_{ij}\, x_i\, x_j = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

De transversaal van het punt $P(\xi_i)$ over V_1 en V_2 heeft blijkbaar de vergelijkingen

$$x_1 : x_2 : x_3 : x_4 : x_5 : x_6 = \xi_1 : \xi_2 : \xi_3 : \lambda\xi_4 : \lambda\xi_5 : \lambda\xi_6 \quad . \quad . \quad . \quad (3)$$

waarin λ een parameter is. De snijpunten van deze transversaal met Ω worden bepaald door

$$\left.\begin{array}{r} a_{11}\,\xi_1^2 + 2\,a_{12}\,\xi_1\,\xi_2 + \ldots a_{33}\,\xi_3^2 + 2\lambda\,(a_{14}\,\xi_1\,\xi_4 + \ldots a_{36}\,\xi_3\,\xi_6) + \\ + \lambda^2\,(a_{33}\,\xi_3^2 + \ldots a_{66}\,\xi_6^2) = 0 \end{array}\right\} . \quad (4)$$

Ligt nu P op Ω, m.a.w. is P het beeld van een rechte in R_3, dus is $\Sigma a_{ij}\,\xi_i\,\xi_j = 0$, zodat $\lambda = 1$ een wortel is van (4), dan geldt voor de andere wortel

$$\lambda' = \frac{a_{11}\,\xi_1^2 + 2\,a_{12}\,\xi_1\,\xi_2 + \ldots a_{33}\,\xi_3^2}{a_{44}\,\xi_4^2 + 2\,a_{45}\,\xi_4\,\xi_5 + \ldots a_{66}\,\xi_6^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Stelt men dus

$$\left.\begin{array}{l} Q_1\,(x_1\,x_2\,x_3) \equiv a_{11}\,x_1^2 + 2\,a_{12}\,x_1\,x_2 + \ldots a_{33}\,x_3^2 \\ Q_2\,(x_4\,x_5\,x_6) \equiv a_{44}\,x_4^2 + 2\,a_{45}\,x_4\,x_5 + \ldots a_{66}\,x_6^2 \end{array}\right\} \quad . \quad . \quad . \quad (6)$$

dan volgt uit (3) en (5), dat de involutorische verwantschap tussen de rechten l en l', met de coördinaten x_i en x'_i, wordt uitgedrukt door de vergelijkingen

$$\left.\begin{array}{l} x_1' = x_1 \,.\, Q_2\,(x_4, x_5, x_6) \;,\; x_4' = x_4 \,.\, Q_1\,(x_1, x_2, x_3) \\ x_2' = x_2 \,.\, Q_2\,(x_4, x_5, x_6) \;,\; x_5' = x_5 \,.\, Q_1\,(x_1, x_2, x_3) \\ x_3' = x_3 \,.\, Q_2\,(x_4, x_5, x_6) \;,\; x_6' = x_6 \,.\, Q_1\,(x_1, x_2, x_3) \end{array}\right\} . \quad . \quad . \quad (7)$$

Zij is dus van de derde graad. Uit (7) leidt men af, dat singulier zijn voor deze verwantschap de rechten van de congruentie C met de vergelijkingen $Q_1 = Q_2 = 0$; aan een dergelijke rechte zijn toegevoegd zulke waarvoor geldt:

$$x_1' : x_2' : x_3' : x_4' : x_5' : x_6' = x_1 : x_2 : x_3 : \lambda x_4 : \lambda x_5 : \lambda x_6,$$

dus de rechten van een waaier, die tot C behoort. Als doorsnede van twee quadratische complexen is C een congruentie (4,4). Tot C behoren de regelscharen S_1 en S_2, die resp. door $x_4 = x_5 = x_6 = Q_1 = 0$ en $x_1 = x_2 = x_3 = Q_2 = 0$ worden voorgesteld. De rechten van zo'n regelschaar zijn hoofdstralen van de verwantschap: aan elk is een (lineaire) congruentie toegevoegd en wel b.v. aan de rechte $(\xi_1 \xi_2 \xi_3 000)$, waarvoor $Q_1 (\xi_1 \xi_2 \xi_3) = 0$ is, de congruentie met de vergelijkingen $x_1 : x_2 : x_3 = \xi_1 : \xi_2 : \xi_3$. De meetkundige plaats van deze congruenties is het quadratisch complex $Q_1 = 0$. In de vijfdimensionale beeldruimte komen met de regelscharen S_1 en S_2 twee kegelsneden K_1 en K_2 overeen, die de doorsnede vormen van Ω met V_1 resp. V_2. De quadratische complexen $Q_1 = 0$ en $Q_2 = 0$ corresponderen met (de doorsnede van Ω met) twee kegels, die V_2 tot toppenvlak en K_1 tot richtkromme, resp. V_1 tot toppenvlak en K_2 tot richtkromme hebben. Hun doorsnede is een $V_3{}^4$ en de doorsnede V daarvan met Ω een $V_2{}^8$, die met de congruentie C correspondeert en die de beschrijvende vlakken van beide stelsels op Ω elk vier maal snijdt. K_1 en K_2 zijn dubbelkrommen van V, door elk punt gaan twee beschrijvenden en men kan V voortgebracht denken door de verbindingslijnen van toegevoegde punten van een tussen K_1 en K_2 met behulp van Ω bepaalde (2,2) verwantschap.

§ 4. Daar de involutie van de 3e graad is, zal in R_5 aan een rechte een cubische kromme, dus in R_3 aan een waaier een cubisch regeloppervlak zijn toegevoegd en in het algemeen aan een regeloppervlak van de graad n een regeloppervlak van de graad $3n$. Het is echter duidelijk, dat bij bijzondere ligging ten opzichte van de singuliere elementen dit antwoord moet worden gewijzigd. Heeft in R_5 een op Ω gelegen kromme een, niet op K_1 of K_2 gelegen snijpunt met V, dan wordt door de involutie aan dit punt een rechte m van V toegevoegd, door welke omstandigheid de graad van de toegevoegde kromme dus met één wordt verminderd. Uit continuïteitsoverwegingen is voorts duidelijk, dat de toegevoegde kromme, na weglating van m, met deze rechte en dus met V een punt gemeen heeft.

Beschouwen wij nu een op Ω gelegen rechte, die K_1 snijdt in het punt P. Daarmede correspondeert in R_3 een waaier, die een rechte p van S_1 bevat. Aan p is in de involutie toegevoegd de lineaire congruentie, die tot richtlijnen heeft de rechten van R_2 die door p worden gesneden. Doorloopt men de waaier waartoe p behoort en die tot top heeft het op p gelegen punt T en die ligt in het vlak U door P, dan snijden de van p verschillende rechten van de waaier alle dezelfde twee rechten a_1 en a_2 van R_1, nl. die door het

punt T en die in het vlak U. De rechten van de waaier $T(U)$ snijden de rechten van de regelschaar R_2 volgens een quadratische involutie. Volgens een van CHASLES afkomstige stelling vormen de transversalen over de paren toegevoegde rechten van een regelschaar een lineair complex. De in onze involutie aan de stralen van de waaier $T(U)$ toegevoegde rechten liggen in dit complex en behoren tot de congruentie met de richtlijnen a_1 en a_2. De doorsnede van het complex en de congruentie is een regelschaar, waartoe echter de waaier $T(U)$ zelf behoort. Hieruit volgt, dat aan de waaier $T(U)$ — afziende van de aan p toegevoegde singuliere congruentie — door de involutie een waaier toegevoegd wordt (en wel een waaier, waarvan de top op a_2 ligt en het vlak door a_1 gaat). De omstandigheid, dat in R_5 de op Ω gelegen rechte de kegelsnede K_1 snijdt, vermindert dus de graad der toegevoegde figuur met *twee*.

Tenslotte maken wij nog de opmerking, dat door de involutie aan een op de kegel $Q_1 = 0$, maar niet op V gelegen punt ten duidelijkste een punt van K_1 is toegevoegd. Wij zijn nu in staat voor een willekeurige kromme op Ω de graad der toegevoegde te bepalen. Is de gegeven kromme γ van de graad n en heeft zij a snijpunten met K_1, b snijpunten met K_2 en nog c snijpunten met V, die niet op K_1 of K_2 liggen, dan geldt voor de graad n' van de toegevoegde kromme γ':

$$n' = 3n - 2a - 2b - c. \quad \quad (8^a)$$

De gegeven kromme heeft $2n$ snijpunten met $Q_1 = 0$; hiervan liggen er a op K_1 en b op K_2, welke laatste dubbelpunten van de kegel zijn en voorts nog c andere op V. Er zijn dus $2n - a - 2b - c$ niet op V gelegen snijpunten met $Q_1 = 0$, zodat voor het aantal snijpunten (a') van γ' met K_1 geldt

$$a' = 2n - a - 2b - c. \quad \quad (8^b)$$

Evenzo is

$$b' = 2n - 2a - b - c \quad \quad (8^c)$$

en ten slotte

$$c' = c. \quad \quad (8^d)$$

In de lijnenruimte heeft men dus: *Aan een regeloppervlak van de graad n, die a rechten gemeen heeft met de regelschaar S_1, b met de regelschaar S_2 en nog c andere met de singuliere congruentie C, wordt door de involutie toegevoegd een regeloppervlak, waarvan de vier overeenkomstige getallen n', a', b' en c' bepaald worden door de betrekkingen (8).*

§ 5. Wij zullen een overeenkomstige eigenschap afleiden voor toegevoegde *complexen*. Uit (7) volgt onmiddellijk, dat aan een complex van de $n$de graad in het algemeen een complex van de graad $3n$ is toegevoegd. Dit resultaat zal wijziging ondergaan, als het gegeven complex bijzondere ligging heeft ten opzichte van de figuur der singulariteiten. Als een complex van de graad n de congruentie C m-voudig bevat en daarenboven de regel-

schaar S_1 p_1-voudig en S_2 p_2-voudig; dan heeft de vergelijking de volgende gedaante

$$f_0 Q_1^m + f_1 Q_1^{m-1} Q_2 + f_2 Q_1^{m-2} Q_2^2 + \ldots + f_m Q_2^m = 0 \quad . \quad . \quad (9)$$

waarbij f_i een polynoom van de graad $n-2m$ is, dat in x_4, x_5, x_6 van de graad $n-m-p_1-2i$ en in x_1, x_2, x_3 van de graad $n-m-p_2-2(m-i)$ is; dat wil zeggen, f_i is in x_1, x_2, x_3 van de graad p_1+2i-m en in x_4, x_5, x_6 van de graad p_2+m-2i.

Voeren wij nu de transformatie (7) uit, dan gaat (9) over in een vergelijking van de graad $3n$. Na de substitutie bevat f_i de factor

$$Q_1^{p_2+m-2i} \; Q_2^{p_1+2i-m}$$

en daar Q_1 overgaat in $Q_1 Q_2^2$ en Q_2 in $Q_1^2 Q_2$, bevat $f_i \; Q_1^{m-i} Q_2^i$ na de transformatie de factor

$$Q_1^{p_2+2m-i} \; Q_2^{p_1+m+i},$$

waaruit volgt, dat het linkerlid van (9) de factor

$$Q_1^{p_2+m} \; Q_2^{p_1+m}$$

bevat. Hieruit volgt, dat het toegevoegde complex na terzijdestelling van de quadratische complexen $Q_1=0$ en $Q_2=0$ de graad

$$n' = 3n - 4m - 2p_1 - 2p_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10^a)$$

bezit. Na de substitutie van (7) in (9) wordt f_i van de graad $n-2m$ in de uitdrukkingen Q_1 en Q_2, terwijl $Q_1^{m-i}\,Q_2^i$ overgaat in $Q_1^{m+i} Q_2^{2m-i}$, zodat afgezien van de factor $Q_1^{p_2+m}\,Q_2^{p_1+m}$ een polynoom in Q_1 en Q_2 ontstaat van de graad

$$(n-2m) + (m+i) + (2m-i) - (p_2+m) - (p_1+m),$$

waaruit volgt

$$m' = n - m - p_1 - p_2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10^b)$$

Die term van f_i, die in x_1, x_2, x_3 de hoogste graad heeft, is van het type

$$x_1^{p_1+2i-m} \; x_4^{n-m-p_1-2i};$$

de uitdrukking

$$x_1^{p_1+2i-m} \,.\, x_4^{n-m-p_1-2i} \; Q_1^{m-i} \; Q_2^i$$

gaat door (7) over in

$$x_1^{p_1+2i-m} \,.\, x_4^{n-m-p_1-2i} \; Q_1^{n-p_1-i} \; Q_2^{p_1+m+i},$$

dus na deling door

$$Q_1^{p_2+m} \; Q_2^{p_1+m} \text{ in } x_1^{p_1+2i-m} \,.\, x_4^{n-m-p_1-2i} \; Q_1^{m-i} \; Q_2^i .$$

Hieruit volgt, dat

$$p_1 + 2i - m = p_1' + 2i - m',$$

dus

$$p_1' = n - 2m - p_2 \,. \quad (10^c)$$

en evenzo

$$p_2' = n - 2m - p_1 . \quad (10^d)$$

Wij hebben dus: *aan een complex van de graad n, dat de singuliere congruentie C m maal bevat en bovendien de regelscharen S_1 en S_2 nog respectievelijk p_1 en p_2 maal, wordt door de involutie toegevoegd een complex, waarvoor de overeenkomstige karakteristieke getallen n', m', p_1' en p_2' bepaald zijn door (10).*

§ 6. In § 2 is melding gemaakt van de meetkundig afgeleide stelling, dat de aan zich zelf toegevoegde rechten der involutie een *quadratisch complex* vormen. Wij kunnen dit complex Γ blijkbaar definiëren als *de meetkundige plaats der rechten, die de regelscharen R_1 en R_2 snijden in een parabolisch lijnenviertal*, d.w.z. in een viertal met twee samengevallen transversalen.

In de vijfdimensionale beeldruimte ontstaat het beeld Γ' van Γ als de meetkundige plaats van de raakpunten met Ω van de rechten, die de vlakken V_1 en V_2 snijden en aan Ω raken. Zo opgevat is het probleem een uitbreiding van het in de beschrijvende meetkunde der regeloppervlakken welbekende vraagstuk, de meetkundige plaats te bepalen van de raakpunten der rechten, die twee gegeven rechten snijden en aan een quadratisch oppervlak raken, welke meetkundige plaats een biquadratische kromme van de eerste soort blijkt te zijn [3]). De algemene opgave: in R_{2n+1} de meetkundige plaats te bepalen van de raakpunten der rechten, die twee gegeven ruimten R_n snijden en aan een V_{2n}^2 raken, is gesteld en meetkundig opgelost door BONE [4]); de bedoelde figuur blijkt de doorsnede te zijn van twee quadratische variëteiten. Voor $n = 2$, dus in R_5, blijkt dit onmiddellijk uit onze vergelijkingen (7); immers de dekpunten der involutorische toevoeging zijn de punten van het quadratisch complex Γ met de vergelijking

$$Q_1 = Q_2 . \quad (11)$$

Ook voor willekeurige waarden van n voert een analytische behandeling hier dadelijk tot het resultaat. En zij stelt ons tevens in staat de vraag te beantwoorden naar de aard van de biquadratische variëteit, een opgave waarbij de meetkundige methode, zoals ook uit het antwoord moge blijken, minder kans op succes schijnt te bieden.

Om de gedachten te bepalen beschouwen wij het geval $n = 2$, d.w.z. wij onderzoeken tot welk type het bedoelde quadratische complex der dubbelrechten van de involutie behoort.

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3) Zie b.v. H. J. V. VEEN, Leerboek der Beschrijvende Meetkunde, II, 169 (Groningen 1929); de stelling werd reeds in 1839 bekend gemaakt door CHASLES (Corresp. math. XI, no. 50).

4) BONE, Wiskundige Opgaven **13**, 122 (1925).

De vergelijkingen der beide quadratische variëteiten in R_5, waarvan ons de doorsnede interesseert, zijn

$$\sum_6 a_{ij}\, x_i\, x_j = 0 \text{ en } \sum_{1,2,3} a_{ij}\, x_i\, x_j = \sum_{4,5,6} a_{ij}\, x_i\, x_j \quad . \; . \; . \quad (12)$$

zodat de λ-matrix der uit hen gevormde bundel er als volgt uitziet

$$\left\| \begin{matrix} a_{11}(1-\lambda) & a_{12}(1-\lambda) & a_{13}(1-\lambda) & a_{14} & a_{15} & a_{16} \\ a_{21}(1-\lambda) & a_{22}(1-\lambda) & a_{23}(1-\lambda) & a_{24} & a_{25} & a_{26} \\ a_{31}(1-\lambda) & a_{32}(1-\lambda) & a_{33}(1-\lambda) & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44}(1+\lambda) & a_{45}(1+\lambda) & a_{46}(1+\lambda) \\ a_{51} & a_{52} & a_{53} & a_{54}(1+\lambda) & a_{55}(1+\lambda) & a_{56}(1+\lambda) \\ a_{61} & a_{62} & a_{63} & a_{64}(1+\lambda) & a_{65}(1+\lambda) & a_{66}(1+\lambda) \end{matrix} \right\| \quad (13)$$

Ontwikkelt men de determinant der matrix, dan blijkt onmiddellijk dat elke term evenveel factoren $(1-\lambda)$ als $(1+\lambda)$ bevat; men ziet dat ook in door b.v. de eerste drie rijen met $(1+\lambda)$ te vermenigvuldigen en de laatste drie kolommen door $(1+\lambda)$ te delen. Hieruit volgt, dat *de λ-vergelijking wortels heeft, die twee aan twee elkaars tegengestelde zijn*. De wortel *nul* komt niet voor, omdat Ω een algemene quadratische variëteit is.

Om verdere eigenschappen der λ-matrix af te leiden, merken wij op, dat wij het coördinatenstelsel slechts zover hebben bepaald, dat aan de vlakken V_1 en V_2 resp. de vergelijkingen $x_4 = x_5 = x_6 = 0$ en $x_1 = x_2 = x_3 = 0$ werden toegekend. Blijkbaar mogen wij dus nog de coördinatengroepen x_1, x_2, x_3 en x_4, x_5, x_6 afzonderlijk lineair transformeren. Wij veronderstellen nu, dat K_1 en K_2 beide niet-ontaarde kegelsneden zijn (dus R_1 en R_2 niet-ontaarde quadratische regelscharen); wij kunnen dan drie hoekpunten van het coördinatensimplex kiezen in de hoekpunten van een pooldriehoek van K_1 resp. van K_2, zodat dus $a_{12} = a_{23} = a_{31} = a_{45} = a_{56} = a_{64} = 0$, terwijl a_{11}, a_{22}, a_{33} en a_{44}, a_{55}, a_{66} ongelijk aan nul zijn.

Door geschikte keuze van het eenheidspunt wordt dan verder verkregen dat $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = a_{66}$ en de λ-matrix luidt

$$\left\| \begin{matrix} 1-\lambda & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 1-\lambda & 0 & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 1-\lambda & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & 1+\lambda & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & 1+\lambda & 0 \\ a_{61} & a_{62} & a_{63} & 0 & 0 & 1+\lambda \end{matrix} \right\| \quad . \; . \; . \quad (14)$$

Stelt men de determinant van de derde orde, gevormd door de a_{ij}, gelijk aan D en is A_{ij} in deze determinant de minor van a_{ij}, dan krijgt men na ontwikkeling de volgende λ-vergelijking:

$$(1-\lambda^2)^3 - (1-\lambda^2)^2 \,.\, \Sigma a_{ij}^2 + (1-\lambda^2) \,.\, \Sigma A_{ij}^2 - D^2 = 0 \quad . \; . \; . \quad (15)$$

Deze cubische vergelijking in $1-\lambda^2$ heeft in het algemeen drie verschillende wortels; men kan ook zonder moeite de getallen a_{ij} zó kiezen, dat (15) voorgeschreven wortels heeft. Immers neemt men

$$a_{14}=\sqrt{p_1},\ a_{25}=\sqrt{p_2},\ a_{36}=\sqrt{p_3},\ a_{15}=a_{16}=a_{24}=a_{26}=a_{34}=a_{35}=0,$$

dan wordt (15):

$$(1-\lambda^2)^3-(1-\lambda^2)(p_1+p_2+p_3)+(1-\lambda^2)(p_2 p_3+p_3 p_1+p_1 p_2)-p_1 p_2 p_3=0,$$

zodat men voor $1-\lambda^2$ resp. vindt p_1, p_2 en p_3.

We komen dus tot de volgende conclusie. De 6 wortels der λ-vergelijking van het quadratisch complex zijn in het algemeen verschillend; ze zijn twee aan twee elkaars tegengestelde, maar kunnen overigens willekeurige waarden aannemen.

Meetkundig wil dat zeggen: het complex Γ der dubbelstralen is in het algemeen van het algemene type; het heeft in de classificatie van KLEIN de Segre-notatie [1 1 1 1 1 1]. Maar Γ is geenszins het algemene quadratische complex, want de zes (verschillende) wortels λ_i der λ-vergelijking voldoen aan de relaties $\lambda_1+\lambda_4=0$, $\lambda_2+\lambda_5=0$, $\lambda_3+\lambda_6=0$. Aangezien voor de eigenschappen van Γ niet deze wortels zelf beslissend zijn, maar λ nog aan een lineaire transformatie onderworpen mag worden, komt het hierop neer, dat Γ één projectief-invariante merkwaardigheid vertoont. Men kan deze in R_5 ook zo formuleren: door Γ'', die de doorsnede is van Ω met een andere quadratische variëteit Q, gaan zes hyperkegels, die men zo kan groeperen, dat zij drie paren vormen van een involutie in de bundel (Ω, Q).

Overeenkomstige eigenschappen gelden voor het bovengenoemde probleem in R_{2n+1}. Voor $n=1$ krijgen wij een λ-vergelijking met vier verschillende wortels, die twee aan twee elkaars tegengestelde zijn. Daar elk viertal verschillende getallen door lineaire transformatie in een dergelijk viertal kan worden overgevoerd, komen wij tot de conclusie, dat de biquadratische kromme van de eerste soort, die als contactkromme in de theorie der conoïden optreedt, projectief de algemene kromme is.

Wat het complex der dubbelstralen betreft, hierbij zijn uiteraard een groot aantal bijzondere gevallen mogelijk, die door de discussie van de elementaire delers van (14) kunnen worden gevonden en corresponderen met bijzondere liggingen van de gegeven regelscharen R_1 en R_2. Wij noemen alleen het meest gespecialiseerde geval:

$a_{ij}=0$ voor alle betrokken indices. De λ-vergelijking luidt dan $(1-\lambda^2)^3=0$ en de Segre-notatie is [(1 1 1) (1 1 1)]. Men gaat gemakkelijk na, dat V_1 en V_2 geconjugeerde vlakken zijn t.o.v. Ω, zodat R_1 en R_2 op dezelfde hyperboloïde H liggen en in de involutie toegevoegde rechten niets anders zijn dan wederkerige poolrechten ten opzichte van H.

Mathematics. — *Over het assenoppervlak van een bundel lineaire complexen.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 27, 1945.)

1. De meetkundige plaats der assen van de exemplaren van een bundel lineaire complexen is in de Euclidische meetkunde een regeloppervlak van de derde graad: de *cylindroïde* van CAYLEY. Volgens een reeds van CLEBSCH afkomstige stelling is de overeenkomstige figuur bij niet-Euclidische maatbepaling een regeloppervlak O van de vierde graad met twee richtlijnen [1]). In het volgende onderzoeken wij in hoeverre een merkwaardige, door APPELL gevonden eigenschap van de cylindroïde ook voor dit oppervlak geldt. Wij bepalen ons kortheidshalve tot de elliptische meetkunde.

2. Is Ω het absolute kwadratisch oppervlak, dan zal de collineatie die het product is van de poolcorrelatie t.o.v. Ω en de met een gegeven lineair complex verbonden nulcorrelatie in het algemeen een dektetraëder bezitten, waarvan de hoekpunten op Ω liggen en de vlakken aan Ω raken. Vier der ribben zijn beschrijvende lijnen van Ω, de twee overige ribben zijn de beide rechten, die zowel in de poolcorrelatie als in de nulcorrelatie aan elkaar zijn toegevoegd; het zijn de *assen* van het lineair complex ten aanzien van de door Ω vastgelegde maatdefinitie.

Een bundel lineaire complexen bevat twee, in het algemeen verschillende, speciale lineaire complexen. Zijn d_1 en d_2 hun dragers, l en l' de (reële) gemeenschappelijke loodlijnen van d_1 en d_2, A_1 en A'_1 de snijpunten van d_1 met l en l', A_2 en A'_2 die van d_2 met l en l', dan zijn de middens M_1 en M_2 van A_1 en A_2 en de middens M'_1 en M'_2 van A'_1 en A'_2 de hoekpunten van een poolviervlak van Ω. Wij kiezen dit tot coördinatentetraëder en wel zo, dat $M_1 M'_1$ en $M_2 M'_2$ de vergelijkingen $x_1 = x_2 = 0$ en $x_3 = x_4 = 0$ krijgen, terwijl Ω door $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ wordt voorgesteld. De afstanden $A_1 A_2$ en $A'_1 A'_2$ zijn resp. $\psi = 2\varphi$ en $\psi' = 2\varphi'$; men doet de algemeenheid niet te kort door de veronderstellingen

$$0 < \psi < \frac{\pi}{2}, \; 0 < \psi' < \frac{\pi}{2}, \; \psi' < \psi.$$

Voor $\psi' = \psi$ zouden d_1 en d_2 Clifford-parallel zijn.

Het speciale lineaire complex met d_1 tot drager heeft nu de vergelijking

$$\sin\varphi \sin\varphi' \, p_{13} - \sin\varphi \cos\varphi' \, p_{14} - \cos\varphi \sin\varphi' \, p_{23} + \cos\varphi \cos\varphi' \, p_{24} = 0 \quad (1)$$

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[1]) CLEBSCH–LINDEMANN, Vorlesungen über Geometrie II (Leipzig 1891), 343—356.

en de bundel lineaire complexen:

$$ap_{13}+bp_{14}+cp_{23}+dp_{24}=0 \quad \ldots \ldots \quad (2)$$

waarbij

$$\left.\begin{array}{l} a=(1+\mu)\sin\varphi\sin\varphi', \quad b=(1-\mu)\sin\varphi\cos\varphi', \\ c=(1-\mu)\cos\varphi\sin\varphi', \quad d=(1+\mu)\cos\varphi\cos\varphi' \end{array}\right\} \quad \ldots \quad (3)$$

en μ de bundelparameter is.

Het product van de pool- en de nulcorrelatie is de collineatie

$$x_1^1=ax_3+bx_4, \; x_2^1=cx_3+dx_4, \; x_3^1=-ax_1-cx_2, \; x_4^1=-bx_1-dx_2 \quad (4)$$

met de karakteristieke vergelijking

$$\lambda^4+(a^2+b^2+c^2+d^2)\lambda^2+(ad-bc)^2=0 \quad \ldots \quad (5)$$

met vier zuiver imaginaire wortels, die twee aan twee elkaars tegengestelde zijn. Is λ_1 een wortel, dan vindt men voor de coördinaten van het bijbehorende dekpunt:

$$\left.\begin{array}{l} x_1:x_2:x_3:x_4=b\lambda_1^2-c\,(ad-bc):d\lambda^2+ \\ +a\,(ad-bc):-\lambda_1\,(ab+cd):\lambda_1\,(a^2+c^2+\lambda_1^2) \end{array}\right\} \quad \ldots \quad (6)$$

dat op Ω ligt. De verbindingslijn met het toegevoegd imaginaire dekpunt snijdt, zoals men dadelijk inziet, zowel $x_3=x_4=0$ als $x_1=x_2=0$. De meetkundige plaats der assen is dus een regeloppervlak met l en l' tot richtlijnen. Voor *de vergelijking van het oppervlak* O vindt men door eliminatie van μ na een korte berekening

$$\sin\psi\,.\,(x_1^2+x_2^2)\,x_3\,x_4-\sin\psi'\,.\,(x_3^2+x_4^2)\,x_1\,x_2=0. \quad \ldots \quad (7)$$

3. Een regeloppervlak van de vierde graad met twee richtlijnen heeft het geslacht één. Wij leiden voor O een parametervoorstelling af met behulp van elliptische functies en stellen daarbij $k=\frac{\sin\psi'}{\sin\psi}$, zodat $0<k<1$ is. Voor

$$y_1:y_2:y_3=x_1\,x_4:x_2\,x_3:x_2\,x_4 \quad \ldots \ldots \quad (8)$$

gaat (7) over in

$$(y_1^2+y_3^2)\,y_2-k\,(y_2^2+y_3^2)\,y_1=0 \quad \ldots \ldots \quad (9)$$

welke vergelijking men interpreteren kan als die van een vlakke cubische kromme. Door de lineaire transformatie

$$y_1=k^2\,z_1-z_2, \; y_2=k\,(z_1-z_2), \; y_3=z_3 \quad \ldots \quad (10)$$

ontstaat

$$z_1\,z_3^2-z_2\,(z_2-z_1)\,(z_2-k^2\,z_1)=0 \quad \ldots \ldots \quad (11)$$

die bevredigd wordt door

$$z_1:z_2:z_3=t^3:t:\sqrt{(1-t^2)\,(1-k^2\,t^2)}. \quad \ldots \ldots \quad (12)$$

Voert men nu de elliptische functie met modulus k in en stelt men

$$t = sn\,u,$$

dan is

$$\sqrt{1-t^2} = cn\,u\ ,\ \sqrt{1-k^2t^2} = dn\,u, \quad \ldots\ldots \quad (13)$$

zodat als parametervoorstelling van (11) gevonden wordt

$$z_1 : z_2 : z_3 = sn^3\,u : sn\,u : cn\,u\;dn\,u. \quad \ldots\ldots \quad (14)$$

Voor het oppervlak (7) vindt men dan de volgende voorstelling met behulp van twee parameters u en v:

$$x_1 = \frac{sn\,u\,.\,dn\,u}{cn\,u}\,v,\ x_2 = v,\ x_3 = \frac{k\,sn\,u\,.\,cn\,u}{dn\,u},\ x_4 = 1. \quad \ldots \quad (15)$$

Elke waarde van u wijst een beschrijvende van het oppervlak aan. Daar de in (15) voorkomende functies van u de perioden $2\,K$ en $2\,i\,K'$ hebben, worden de beschrijvenden afgebeeld op de punten van een periodenrechthoek met zijden $2\,K$ en $2\,i\,K'$ en wel, zoals men gemakkelijk nagaat, *reëele* beschrijvenden op die punten, waarvoor geldt: òf u is reëel òf $u - i\,K'$ is reëel. Wij zien er van af de welbekende projectieve eigenschappen van het oppervlak aan de hand van (15) af te leiden.

4. De cylindroïde heeft de volgende merkwaardige eigenschap: *de voetpuntskromme van een willekeurig punt der ruimte ten opzichte van de beschrijvenden, is een vlakke kromme* (en wel een ellips). APPELL heeft bewezen dat, afziende van het triviale geval der cylinderoppervlakken, de cylindroïde het enige regeloppervlak is met de genoemde eigenschap [2]).

Het ligt voor de hand na te gaan of aan het oppervlak O in de elliptische ruimte de overeenkomstige eigenschap toekomt.

Wij maken daartoe gebruik van de parametervoorstelling (15) der beschrijvenden:

$$\frac{x_1}{x_2} = \frac{sn\,u\,.\,dn\,u}{cn\,u},\qquad \frac{x_3}{x_4} = \frac{k\,sn\,u\,.\,cn\,u}{dn\,u}. \quad \ldots\ldots \quad (16)$$

De poolrechte t.o.v. Ω heeft de vergelijkingen

$$\frac{x_1}{x_2} = -\frac{cn\,u}{sn\,u\,.\,dn\,u},\qquad \frac{x_3}{x_4} = -\frac{dn\,u}{k\,sn\,u\,.\,cn\,u} \quad \ldots \quad (17)$$

zodat de vergelijking van het vlak door deze rechte en het willekeurige punt $P \equiv (y_1\,y_2\,y_3\,y_4)$ luidt:

$$\left.\begin{array}{r}(k\,sn\,u\,.\,cn\,u\,.\,y_3 + dn\,u\,.\,y_4)\,(sn\,u\,.\,dn\,u\,.\,x_1 + cn\,u\,.\,x_2) - \\ -(sn\,u\,.\,dn\,u\,.\,y_1 + cn\,u\,.\,y_2)\,.\,(k\,sn\,u\,.\,cn\,u\,.\,x_3 + dn\,u\,.\,x_4) = 0.\end{array}\right\} \quad (18)$$

[2]) APPELL. Propriété caractéristique du cilindroïde, Bull. Soc. Math., **28** (1900) 261—265. Vgl. ook BRICARD, id. **29** (1901), 18—21; DEMOULIN, id. 30—50; SOMMER, Arch. Math. und Ph. **20**, 311—320.

Het snijpunt van dit vlak met (16) d.w.z. de projectie van P op de beschrijvende, heeft dus de coördinaten:

$$\left.\begin{aligned} x_1 &= sn\,u \,.\, dn\,u \,.\, (sn\,u \,.\, dn\,u \,.\, y_1 + cn\,u \,.\, y_2), \\ x_2 &= cn\,u \,.\, (sn\,u \,.\, dn\,u \,.\, y_1 + cn\,u \,.\, y_2), \\ x_3 &= k \,.\, sn\,u \,.\, cn\,u \,.\, (k \,.\, sn\,u \,.\, cn\,u \,.\, y_3 + dn\,u \,.\, y_4), \\ x_4 &= dn\,u \,.\, (k\, sn\,u \,.\, cn\,u \,.\, y_3 + dn\,u \,.\, y_4). \end{aligned}\right\} \quad . \; . \; . \quad (19)$$

Om na te gaan of dit voetpunt, bij variabele waarde van u, een vlakke kromme doorloopt, substitueren wij (19) in

$$A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4 = 0$$

en verkrijgen dan een vergelijking van de gedaante

$$P_1 + P_2\, sn^2\, u + P_3\, sn^4\, u + P_4\, sn\,u \,.\, cn\,u \,.\, dn\,u = 0,$$

welke blijkbaar alleen dan voor elke waarde van u bevredigd wordt als $P_1 = P_2 = P_3 = P_4 = 0$, zodat voor A_i de volgende vergelijkingen ontstaan

$$\left.\begin{aligned} A_1 y_2 + A_2 y_1 + k A_3 y_4 + k A_4 y_3 &= 0 \\ A_2 y_2 \qquad\qquad + A_4 y_4 &= 0 \\ A_1 y_1 - A_2 y_2 + k^2 A_3 y_3 - k^2 A_4 y_4 &= 0 \\ A_1 y_1 \qquad + A_3 y_3 \qquad\qquad &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \quad (20)$$

die alleen een van nul verschillende oplossing hebben als

$$(y_1^2 + y_2^2)\, y_3\, y_4 - k\,(y_3^2 + y_4^2)\, y_1\, y_2 = 0.$$

Wij hebben dus: *de voetpuntskromme van een punt P t.o.v. de beschrijvenden van het regeloppervlak O is dan en alleen dan een vlakke kromme, als P op O ligt.*

Uit (20) volgt, dat de vergelijking van het vlak der voetpuntskromme luidt:

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} = \frac{x_3}{y_3} + \frac{x_4}{y_4}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

Dit vlak gaat uiteraard door P. Tenzij P op Ω of op l of l' ligt, bevat het vlak geen richtlijn of beschrijvende van O, dat wil dus zeggen, dat in het algemeen *de voetpuntskromme een vlakke kromme van de vierde graad* is met twee dubbelpunten.

Mathematics. — *An arithmetical property of a class of* DIRICHLET'*s series.* By J. POPKEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 27, 1945.)

A convergent DIRICHLET's series

$$D(s) = \sum_{h=1}^{\infty} a_h e^{-\lambda_h s} \; (\lambda_1 < \lambda_2 < \lambda_3 < \ldots; \lim_{h\to\infty} \lambda_h = +\infty)$$

represents an analytic function $f(s)$. In this paper we suppose that $f(s)$ is an algebraic-transcendental function, i.e. $f(s)$ satisfies an algebraic differential equation

$$F(s, f(s), f'(s), \ldots, f^{(n)}(s)) = 0,$$

$F(s, f, f', \ldots, f^{(n)})$ being a polynomial in $s, f, f', \ldots f^{(n)}$. An extensive investigation with respect to this class of DIRICHLET's series was made by OSTROWSKI in his dissertation [1]).

Here we consider only series with the additional property that all coefficients a_h are rationals. Writing these coefficients in their irreducible forms, we obtain an upperbound for the largest prime divisor of the denominator of a_h.

Theorem. *Let the convergent* DIRICHLET'*s series*

$$D(s) = \sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$$

have rational coefficients a_h and represent an analytic function satisfying an algebraic differential equation. Let p_h denote the largest prime divisor of the denominator of a_h, where a_h is written in its irreducible form [2]). *Then there exists a positive number c, such that*

$$p_h < \lambda_h^c$$

for all but a finite number of values for h.

We may apply this result on a power series

$$\sum_{h=0}^{\infty} a_h x^h,$$

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[1]) A. OSTROWSKI, Über Dirichletsche Reihen und algebraische Differentialgleichungen. Mathem. Zeitschrift 8, 241—298 (1921).

[2]) If $a_h = 0$, put $a_h = \frac{0}{1}$. In the proof of this theorem (lemma 1) we suppose, that all coefficients a_h are different from zero; it is clear that this means no loss of generality.

convergent in the vincinity of the origin, with rational coefficients a_h and representing an algebraic-transcendental function $f(x)$.

In fact, the substitution $x = e^{-s}$ transforms the power series into a convergent DIRICHLET's series

$$\sum_{h=0}^{\infty} a_h e^{-hs},$$

defining the analytic function $f(e^{-s})$. Now both functions $f(x)$ and e^{-s} are algebraic-transcendentals, hence by a general theorem [3]) the result of the substitution, $f(e^{-s})$, is also an algebraic-transcendental function.

Applying the previous theorem we obtain a well-known theorem of HURWITZ [4]):

Suppose that the convergent power series

$$\sum_{h=0}^{\infty} a_h x^h$$

with rational coefficients a_h, represents an algebraic-transcendental function. Let p_h denote the largest prime divisor in the denominator of a_h, when a_h is written in its irreducible form. Then there exists a positive number c_1, such that

$$p_h < h^{c_1} \quad \text{for} \quad h > 1.$$

Notations: In this paper h always denotes a variable taking all positive integral values, $c, c_1, c_2, \ldots$ denote positive constants and $\gamma_1, \gamma_2, \ldots, \gamma_m$ other constants.

If a relation, such as $p_h < \lambda_h{}^c$, is valid for all but a finite number of values for h, then we use the expression "the relation is valid for sufficiently large values of h", often in the abreviated form "f.s.l. h".

Let $P(x_1, x_2, \ldots, x_n;\ y_1, y_2, \ldots, y_m)$ be a polynomial in the variables $x_1, x_2, \ldots, x_n;\ y_1, y_2, \ldots, y_m$, then we often will use the form $P(x_\nu; y_\mu)$. Similarly $(x_\nu; y_\mu)$ denotes the set of quantities $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$.

A formal sum consisting of zero terms will be equal to zero.

At first we shall state without proof some lemmas necessary for the deduction of our theorem. Then we shall show, that this theorem indeed is a collorary of these lemmas and lastly we shall devellop the missing proofs for all these propositions.

Lemma 1: Let the convergent DIRICHLET's series $\sum\limits_{h=1}^{\infty} a_h e^{-\lambda_h s}$ with positive exponents λ_h represent an algebraic-transcendental function. Let all coefficients a_h be different from zero. Then there exists a positive integer N and a number $\lambda_0 \geqq 0$, such that for sufficiently large values of h any number $\lambda_h + \lambda_0$ is a sum of N exponents at most, taken from the sequence $\lambda_1, \lambda_2, \ldots, \lambda_{h-1}$.

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3) See A. OSTROWSKI, loc. cit. p. 269.

4) A. HURWITZ, Sur le développement des fonctions satisfaisant à une équation différentielle algébrique. Annales de l'Ecole Norm. Sup. (3) 6, 327—332 (1889).

We shall apply this lemma in the following form: There exists a positive integer r, such that

$$\lambda_h = \lambda_{h_1} + \lambda_{h_2} + \ldots + \lambda_{h_t} - \lambda_0 \qquad \text{for } h > r. \quad . \quad . \quad . \quad (1)$$

where $t = t(h)$ denotes a positive integer $\leqq N$ and $h_1, h_2, \ldots, h_t$ represent positive integers $< h$.

It follows $t \geqq 2$, for otherwise $\lambda_h = \lambda_{h1} - \lambda_0 \leqq \lambda_{h1} < \lambda_h$.

More generally we often take $r + 1$ arbitrary numbers $\overline{\lambda}_0, \overline{\lambda}_1, \ldots, \overline{\lambda}_r$, which may be complex. Then we consider the system of recurrent relations

$$\overline{\lambda}_h = \overline{\lambda}_{h_1} + \overline{\lambda}_{h_2} + \ldots + \overline{\lambda}_{h_t} - \overline{\lambda}_0 \qquad \text{for } h > r. \quad . \quad . \quad . \quad (2)$$

where r and — for any given suffix h — the symbols $t, h_1, h_2, \ldots, h_t$ denote the same positive integers as in the formula (1).

It is clear that these relations (2) are equivalent with the system

$$\overline{\lambda}_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \overline{\lambda}_\varrho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the coefficients $g_{h\varrho}$ are found in the following manner:

For

$$h = 1, 2, \ldots, r \text{ let } g_{h\varrho} = \begin{cases} 1 \text{ if } h = \varrho, \\ 0 \text{ if } h \neq \varrho; \end{cases}$$

and for $h > r$ we define by induction

$$g_{h0} = \sum_{\tau=1}^{t} g_{h_\tau 0} - 1,$$

$$g_{h\varrho} = \sum_{\tau=1}^{t} g_{h_\tau \varrho} \qquad (\varrho = 1, 2, \ldots, r).$$

Evidently all coefficients $g_{h\varrho}$ are integers. We obtain in particular

$$\lambda_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \lambda_\varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Lemma 2: Let the convergent DIRICHLET's series $\sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$ represent an algebraic-transcendental function. Then there exists a polynomial $u(x) \not\equiv 0$ and a number of constants $\gamma_1, \gamma_2, \ldots \gamma_m$, such that for sufficiently large values of h the quantity $a_h\, u(\lambda_h)$ can be expressed as a polynomial with integral coefficients in $a_1, a_2, \ldots, a_{h-1}$ and in $\gamma_1, \gamma_2, \ldots, \gamma_m$.

In other words: these exists for sufficiently large h a polynomial $f_h(x_1, x_2, \ldots, x_{h-1}; z_1, z_2, \ldots, z_m)$ with integral coefficients, such that

$$a_h\, u(\lambda_h) = f_h(a_1, a_2, \ldots, a_{h-1}; \gamma_\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Lemma 3: Let r be the positive integer defined in the formula (1). Let the sequence $\overline{\lambda}_0, \overline{\lambda}_1, \overline{\lambda}_2, \ldots$ consist of $r + 1$ arbitrary numbers $\overline{\lambda}_0, \overline{\lambda}_1, \ldots, \overline{\lambda}_r$

and for $h > r$ of quantities $\overline{\lambda}_h$ defined by the relations

$$\overline{\lambda}_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \overline{\lambda}_\varrho . \qquad (3)$$

Then, if all exponents λ_h of the DIRICHLET's series are positive,

$$|\overline{\lambda}_h| < \lambda_h^3 \quad \text{f. s. l. } h.$$

Lemma 4: Let the exponents λ_h of the DIRICHLET's series be positive. If U denotes an arbitrary positive number, then there exists a positive number $\varepsilon = \varepsilon(U)$ with the following property:

Let $\overline{\lambda}_0, \overline{\lambda}_1, \ldots, \overline{\lambda}_r$ represent $r+1$ numbers, such that

$$|\overline{\lambda}_\varrho - \lambda_\varrho| < \varepsilon \qquad \text{for } \varrho = 0, 1, \ldots, r; \qquad (6)$$

if

$$\overline{\lambda}_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \overline{\lambda}_\varrho \qquad \text{for } h > r, \qquad (3)$$

then it follows

$$|\overline{\lambda}_h| > U \quad \text{f. s. l. } h.$$

Lemma 5: Let $P_k(x_1, x_2, \ldots, x_n)$ for $k = 1, 2, \ldots$ denote a sequence of polynomials with rational coefficients, such that the system of equations

$$P_k(x_1, x_2, \ldots, x_n) = 0$$

at least has one solution $(X_1, X_2, \ldots, X_n)$.

If ε is an arbitrary positive number, then this system also has a solution in algebraic numbers: $(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$, such that

$$|\overline{X}_\nu - X_\nu| < \varepsilon \qquad \text{for } \nu = 1, 2, \ldots, n.$$

Proof of the theorem.

It is clear that all but a finite number of exponents λ_h in the series $D(s)$ must be positive, since λ_h tends to infinity. By subtracting from $D(s)$ a finite sum $\sum_{h=1}^{k} a_h e^{-\lambda_h s}$ we obtain a new series

$$D^*(s) = \sum_{h=1}^{\infty} a_{k+h}\, e^{-\lambda_{k+h} s}$$

with only positive exponents, representing the function

$$f^*(s) = f(s) - \sum_{h=1}^{k} a_h\, e^{-\lambda_h s}.$$

All functions $f(s), e^{-\lambda_h s}$ $(h = 1, 2, \ldots, k)$ are algebraic-transcendentals,

and hence by a known theorem [5]) $f^*(s)$ itself is a function of the same class.

Now let us suppose, that our theorem has been proved in the special case that all exponents λ_h are positive, then we may apply our theorem when $D^*(s)$ is substituted for $D(s)$. We deduce the existence of a positive number c, such that

$$p_{k+h} < \lambda_{k+h}^c$$

holds for all but a finite number of values for h, hence

$$p_{h'} < \lambda_{h'}^c$$

is satisfied for all but a finite number of values for h'. Therefore we may, without loss of generality, suppose that all exponents λ_h in the series $D(s)$ are positive.

Evidently we also may assume that no coefficient a_h is zero.

This series represents an algebraic-transcendental function and therefore we may apply lemma 1. Hence by a collorary

$$\lambda_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \lambda_\varrho, \quad \ldots \ldots \ldots \ldots \quad (4)$$

where r is a fixed positive integer and $g_{h\varrho}$ are integers.

Moreover by lemma 2 there exists a polynomial

$$u(x) = \sum_{\varkappa=0}^{k} \omega_\varkappa x^\varkappa, \qquad \omega_k \neq 0,$$

and f.s.l. h a polynomial $f_h(x_1, x_2, \ldots, x_{h-1}; z_\mu)$ with integral coefficients, such that

$$a_h\, u(\lambda_h) = f_h(a_1, a_2, \ldots, a_{h-1}; \gamma_\mu) \qquad \text{for } h \geqq c_2, \quad \ldots \quad (7)$$

where $c_2 > 0, \gamma_1, \gamma_2, \ldots, \gamma_m$ are appropiately chosen constants. Hence

$$u(\lambda_h) = \sum_{\varkappa=0}^{k} \omega_\varkappa \left(\sum_{\varrho=0}^{r} g_{h\varrho}\, \lambda_\varrho \right)^\varkappa.$$

Defining $U_h = U_h(w_\varkappa; y_\varrho)$ by

$$U_h(w_\varkappa; y_\varrho) = \sum_{\varkappa=0}^{k} w_\varkappa \left(\sum_{\varrho=0}^{r} g_{h\varrho}\, y_\varrho \right)^\varkappa, \quad \ldots \ldots \quad (8)$$

we observe that U_h is a polynomial with integral coefficients in the variables $w_\varkappa; y_\varrho$. It follows

$$u(\lambda_h) = U_h(\omega_\varkappa; \lambda_\varrho)$$

and therefore by (7)

$$a_h\, U_h(\omega_\varkappa; \lambda_\varrho) - f_h(a_1, a_2, \ldots, a_{h-1}; \gamma_\mu) = 0 \qquad \text{for } h \geqq c_2.$$

[5]) "A rational function of algebraic-transcendental functions again is algebraic-transcendental". See A. OSTROWSKI, loc. cit. p. 268 or V. E. E. STADIGH, dissertation, Helsingfors 1902, p. 5.

For $h \geqq c_2$ we define $P_h(w_\varkappa; y_\varrho; z_\mu)$ by

$$a_h U_h(w_\varkappa; y_\varrho) - f_h(a_1, a_2, \ldots, a_{h-1}; z_\mu) = P_h(w_\varkappa; y_\varrho; z_\mu). \quad . \quad . \quad (9)$$

P_h is a polynomial with rational coefficients in $w_\varkappa$; y_ϱ and z_μ. The system of algebraic equations

$$P_h(w_\varkappa; y_\varrho; z_\mu) = 0, \qquad h \geqq c_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

has at least one solution, viz.

$$(\omega_\varkappa; \lambda_\varrho; \gamma_\mu).$$

If ε is an arbitrary positive number, then we deduce from lemma 5, that this same system even has a solution

$$(\overline{\omega}_\varkappa; \overline{\lambda}_\varrho; \overline{\gamma}_\mu)$$

in algebraic numbers, such that

$$|\overline{\omega}_\varkappa - \omega_\varkappa| < \varepsilon, \; |\overline{\lambda}_\varrho - \lambda_\varrho| < \varepsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence, for $h \geqq c_2$, $P_h(\overline{\omega}_\varkappa, \overline{\lambda}_\varrho, \overline{\gamma}_\mu) = 0$; or by (9)

$$a_h U_h(\overline{\omega}_\varkappa, \overline{\lambda}_\varrho) = f_h(a_1, a_2, \ldots, a_{h-1}; \overline{\gamma}_\mu) \qquad \text{for } h \geqq c_2. \quad . \quad . \quad (12)$$

Put

$$\overline{u}(x) = \sum_{\varkappa=0}^{k} \overline{\omega}_\varkappa x^\varkappa$$

and for $h > r$

$$\overline{\lambda}_h = \sum_{\varrho=0}^{r} g_{h\varrho} \overline{\lambda}_\varrho,$$

then we obtain by (8)

$$U_h(\overline{\omega}_\varkappa, \overline{\lambda}_\varrho) = \overline{u}(\overline{\lambda}_h).$$

We observe that the numbers $\overline{\lambda}_0, \overline{\lambda}_1, \overline{\lambda}_2, \ldots$ satisfy the conditions (6) and (3) of lemma 4. As yet ε was an arbitrary positive number. Now we substitute in lemma 4

$$U = 1 + 2\,\frac{|\omega_0| + |\omega_1| + \ldots |\omega_{k-1}| + k}{|\omega_k|}$$

and we choose for $\varepsilon = \varepsilon(U)$ such a small number that it has the property stated in this lemma. Then it follows

$$|\overline{\lambda}_h| > U > 1 \qquad \text{f. s. l. } h.$$

Without loss of generality we may take

$$\varepsilon < 1, \; \varepsilon < \left|\frac{\omega_k}{2}\right|.$$

By means of these inequalities it is easy to prove

$$U_h(\overline{\omega}_\varkappa;\overline{\lambda}_\varrho)=\overline{u}(\overline{\lambda}_h)\neq 0 \qquad \text{f. s. l. h.} \quad \dots \dots \quad (13)$$

In fact; by (11) we have $|\overline{\omega}_\varkappa-\omega_\varkappa|<\varepsilon<1$, hence

$$|\overline{\omega}_\varkappa|<|\omega_\varkappa|+1;$$

further $|\overline{\omega}_k-\omega_k|<\varepsilon<\left|\frac{\omega_k}{2}\right|$, or

$$|\overline{\omega}_k|>\left|\frac{\omega_k}{2}\right|>0.$$

If $k=0$, then $|\overline{u}(\overline{\lambda}_h)|=|\overline{\omega}_k|>0$.

If $k>0$, then f.s.l. h

$$\begin{aligned}|\overline{u}(\overline{\lambda}_h)| &= |\overline{\omega}_k\overline{\lambda}_h^k+\overline{\omega}_{k-1}\overline{\lambda}_h^{k-1}+\dots+\overline{\omega}_0|\\ &\geqq |\overline{\omega}_k\overline{\lambda}_h^k|-(|\overline{\omega}_{k-1}|+\dots+|\overline{\omega}_0|)\,|\overline{\lambda}_h|^{k-1}\\ &> \left|\frac{\omega_k}{2}\right||\overline{\lambda}_h|^k-(|\omega_{k-1}|+\dots+|\omega_0|+k)\,|\overline{\lambda}_h|^{k-1}\\ &= \left|\frac{\omega_k}{2}\right||\overline{\lambda}_h|^{k-1}\left(|\overline{\lambda}_h|-2\,\frac{|\omega_0|+\dots+|\omega_{k-1}|+k}{|\omega_k|}\right)\\ &> \left|\frac{\omega_k}{2}\right||\lambda_h|^{k-1}(|\overline{\lambda}_h|-U)\\ &> 0,\end{aligned}$$

and this completes the proof of (13).

We take a positive integer A such, that all numbers $(A\overline{\omega}_\varkappa, A\overline{\lambda}_\varrho)$ are algebraic integers. Then, taking account of (8), it is clear that

$$A^{k+1}U_h(\overline{\omega}_\varkappa;\overline{\lambda}_\varrho)=\sum_{\varkappa=0}^{k}A\,\overline{\omega}_\varkappa\,.\,A^k\left(\sum_{\varrho=0}^{r}g_{h\varrho}\,\overline{\lambda}_\varrho\right)^\varkappa$$

also are algebraic integers.

We define $\overline{u}_h$ by

$$\overline{u}_h=A^{k+1}U_h(\overline{\omega}_\varkappa;\overline{\lambda}_\varrho),$$

then it follows from (12)

$$a_h\,\overline{u}_h=A^{k+1}f_h(a_1,a_2,\dots,a_{h-1};\overline{\gamma}_\mu), \qquad \text{f. s. l. } h. \quad \dots \quad (14)$$

and from (13)

$$\overline{u}_h\neq 0 \qquad \text{f. s. l. } h.$$

Let

$$\overline{\omega}_\varkappa^{(\tau_\varkappa)}\ (\tau_\varkappa=1,2,\dots,t_\varkappa),\quad \overline{\lambda}_\varrho^{(\sigma_\varrho)} \qquad (\sigma_\varrho=1,2,\dots,s_\varrho)$$

represent the conjugates of the algebraic numbers $\overline{\omega}_{\varkappa}, \overline{\lambda}_{\varrho}$. We addopt the convention

$$\bar{u}_h^{(1)} = \bar{u}_h$$

and consider the $L' = \prod_{\varkappa=0}^{k} \prod_{\varrho=0}^{r} t_{\varkappa} s_{\varrho}$ different forms

$$\bar{u}_h^{(l)} = A^{k+1} U_h (\overline{\omega}_{\varkappa}^{(\tau_{\varkappa})} ; \overline{\lambda}_{\varrho}^{(\sigma_{\varrho})}) \qquad (l = 1, 2, \ldots, L').$$

We remember that the polynomial U_h has integral coefficients, hence all numbers $\bar{u}_h^{(l)}$ are algebraic integers; so that

$$\prod_{l=1}^{L'} (x - \bar{u}_h^{(l)})$$

defines a polynomial in x with rational integers as coefficients. It follows that all conjugate numbers of $\bar{u}_h$ are included in the L' quantities $\bar{u}_h^{(l)}$; we shall denote these conjugates, say, by $\bar{u}_h^{(l)}$ with $l = 1, 2, \ldots, L$ $(L \leqq L')$. Therefore

$$n_h = \prod_{l=1}^{L} \bar{u}_h^{(l)}$$

represents the norm of an algebraic integer $\bar{u}_h$; $\bar{u}_h$ is different from zero f.s.l. h. Hence n_h is a rational integer, not zero f.s.l. h.

Multiplying both sides of (14) with $\prod_{l=2}^{L} \bar{u}_h^{(l)}$ we obtain

$$n_h a_h = A^{k+1} \prod_{l=2}^{L} \bar{u}_h^{(l)} \cdot f_h (a_1, a_2, \ldots, a_{h-1}; \bar{\gamma}_\mu),$$

or

$$n_h a_h = \varphi_h (a_1, a_2, \ldots, a_{h-1}; \bar{\gamma}_\mu) \qquad \text{f. s. l. } h.$$

the right-hand side being a polynomial in $a_1, a_2, \ldots, a_{h-1}$; $\bar{\gamma}_\mu$ with algebraic integers as coefficients.

We write all rationals a_h in their irreducible forms $a_h = \frac{c_h}{d_h}$ $(d_h > 0)$, so that p_h denotes the largest prime divisor of d_h. We take the positive integer B such that all numbers $B\bar{\gamma}_\mu = \bar{\Gamma}_\mu$ are algebraic integers. It follows

$$n_h a_h = \varphi_h \left(\frac{c_1}{d_1}, \frac{c_2}{d_1}, \ldots, \frac{c_{h-1}}{d_{h-1}}; \frac{\bar{\Gamma}_\mu}{B} \right),$$

hence

$$n_h a_h = \frac{\alpha_h}{P_h} \qquad \text{f. s. l. } h,$$

where α_h represents an algebraic integer and P_h denotes a composite positive integer with factors taken from the sequence $B, d_1, d_2, \ldots, d_{h-1}$. Moreover

the algebraic integer $\alpha_h = n_h P_h a_h$ is rational and therefore a rational integer, hence

$$d_h / n_h P_h \rightarrow p_h \mid n_h P_h \rightarrow p_h \mid n_h B d_1 d_2 \dots d_{h-1},$$

or

$$p_h \leqq \max(|n_h|, B, p_1, p_2, \dots, p_{h-1}) \qquad \text{f. s. l. h.} \quad \dots \quad (15)$$

It is easy to determine an upperbound for $|n_h|$: We have by definition

$$\overline{u}_h^{(l)} = A^{k+1} U_h(\overline{\omega}_\varkappa^{(\tau_\varkappa)}; \overline{\lambda}_\varrho^{(\sigma_\varrho)}),$$

hence by (8)

$$\overline{u}_h^{(l)} = A^{k+1} \sum_{\varkappa=0}^{k} \overline{\omega}_\varkappa^{(\tau_\varkappa)} \left(\sum_{\varrho=0}^{r} g_{h\varrho} \overline{\lambda}_\varrho^{(\sigma_\varrho)} \right)^\varkappa \qquad (l = 1, 2, \dots, L).$$

It follows by lemma 3 with $\overline{\lambda}_\varrho^{(\sigma_\varrho)}$ in stead of $\overline{\lambda}_\varrho$ f.s.l. h

$$\left| \sum_{\varrho=0}^{r} g_{h\varrho} \overline{\lambda}_\varrho^{(\sigma_\varrho)} \right| < \lambda_h^3,$$

hence

$$|\overline{u}_h^{(l)}| < A^{k+1} \lambda_h^{3k} \sum_{\varkappa=0}^{k} |\overline{\omega}_\varkappa^{(\tau_\varkappa)}| < \lambda_h^{3k+1} \text{ f. s. l. } h.$$

If we put $c_3 = (3k+1)L$, then it follows

$$|n_h| = \prod_{l=1}^{L} |\overline{u}_h^{(l)}| < \lambda_h^{c_3} \text{ f. s. l. } h. \quad \dots \dots \quad (16)$$

We prove the theorem from this inequality and from (15). Take a positive integer H, such that $\lambda_H > 1$, and that both (15) and (16) are valid for $h \geqq H$; then choose c in such a manner that

$$c \geqq c_3 \text{ and } \lambda_H^c > \max(B, p_1, p_2, \dots, p_H).$$

We shall show by induction that

$$p_h < \lambda_h^c$$

is true for $h \geqq H$.

This inequality evidently is satisfied for $h = H$. Let us suppose that $h > H$ and that the inequality is true for $h-1, h-2, \dots, H$ in stead of h.

From (15) it follows that there are only three possibilities

$1^\circ\ p_h \leqq |n_h|$. $2^\circ\ p_h \leqq \max(B, p_1, p_2, \dots, p_H)$. $3^\circ\ p_h \leqq \max(p_{H+1}, \dots, p_{h-1})$.

If $p_h \leq |n_h|$, then by (16) we obtain $p_h < \lambda_h^{c_3} \leqq \lambda_h^c$ since $\lambda_h > \lambda_H > 1$.

If $p_h \leq \max(B, p_1, p_2, \dots, p_H)$, then it follows $p_h < \lambda_H^c < \lambda_h^c$.

If $p_h \leq \max(p_{H+1}, p_{H+2}, \dots, p_{h-1})$, then we have

$$p_h \leqq \max(\lambda_{H+1}^c, \lambda_{H+2}^c, \dots, \lambda_{h-1}^c) < \lambda_h^c.$$

In all three cases it follows that the inequality also for h is satisfied and this proves the theorem.

In the next section of this paper we shall show that the lemmas we applied are true.

A well-known theorem of OSTROWSKI states, that if a convergent DIRICHLET's series $\sum_{h=1}^{\infty} a_h e^{-\lambda_h s}$ represents an algebraic-transcendental function, then there exists for the set of all exponents λ_h a finite linear basis; i.e. there exists a finite number of exponents $\lambda_1, \lambda_2, \ldots, \lambda_l$, such that all other exponents are expressible as a linear aggregate of $\lambda_1, \lambda_2, \ldots, \lambda_l$ with integral coefficients [6]). OSTROWSKI's reasoning in showing this result may be applied, almost unchanged, to prove that our lemma 1 is true.

A second theorem of OSTROWSKI states that if the DIRICHLET's series represents an algebraic-transcendental function, then there exists a finite number of coefficients $a_1, a_2, \ldots, a_k$, such that all other coefficients a_h may be expressed as rational functions with rational coefficients in $a_1, a_2, \ldots, a_k$ [7]). The proof of this theorem shows that our lemma 2 is valid.

For the deduction of lemma 3 we shall apply lemma 1 in the form of the formula

$$\lambda_h = \sum_{\tau=1}^{t} \lambda_{h_\tau} - \lambda_0 \quad \text{for } h > r. \quad \ldots \ldots \quad (1)$$

Here $t = t(h)$ denotes a positive integer, such that $2 \leqq t \leqq N$ and $h_1, h_2, \ldots, h_t$ represent positive integers $< h$; $\lambda_0 \geqq 0$.

From now on we addopt the convention

$$h_\tau \geqq h_{\tau+1} \qquad (\tau = 1, 2, \ldots, t-1), \quad \ldots \ldots \quad (17)$$

hence

$$\lambda_h > \lambda_{h_\tau} \geqq \lambda_{h_{\tau+1}} > 0.$$

We considered the recurrent relations

$$\bar{\lambda}_h = \sum_{\tau=1}^{t} \bar{\lambda}_{h_\tau} - \bar{\lambda}_0 \qquad \text{for } h > r, \quad \ldots \ldots \quad (2)$$

$\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_r$ being arbitrary numbers and we observed that they were equivalent with the system

$$\bar{\lambda}_h = \sum_{\varrho=0}^{r} g_{h\varrho} \bar{\lambda}_\varrho \quad \ldots \ldots \ldots \quad (3)$$

with integral coefficients $g_{h\varrho}$.

Lemma 3:

$$|\bar{\lambda}_h| < \lambda_h^3 \qquad \text{f. s. l. } h.$$

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6) A. OSTROWSKI, loc. cit., Satz 6, p. 260.

7) The same, loc. cit., Satz 8, p. 262—263.

We divide the proof of this lemma into three parts.

Part I. There exists a positive number c_4, such that

$$\lambda_h - \lambda_{h_1} > c_4 \qquad \text{for } h > r$$

(λ_{h_1} being defined in the formulas (1) and (17)).

Proof: Since λ_h tends to infinity there exists a positive number c_5, such that

$$\lambda_h > \lambda_0 + 1 \qquad \text{for } h \geqq c_5.$$

Now

$$\lambda_h - \lambda_{h_1} = \sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0,\ t \geqq 2 \qquad \text{for } h > r.$$

We consider the two cases $h_2 \geqq c_5$ and $h_2 < c_5$.
If $h_2 \geqq c_5$, then we obtain $\lambda_{h_2} > \lambda_0 + 1$, hence

$$\lambda_h - \lambda_{h_1} \geqq \lambda_{h_2} - \lambda_0 > 1.$$

If, on the other hand, $h_2 < c_5$, then

$$1 \leqq h_\tau \leqq h_2 < c_5 \text{ for } \tau = 3, 4, \ldots, t;\ t \leqq N.$$

There is only a finite number of different sequences $h_2, h_3, \ldots, h_t$ of at most $N - 1$ positive integers, such that each integer is smaller than c_5. Hence in this case $(h_2 < c_5)$ we have only to consider a finite number of different values for

$$\lambda_h - \lambda_{h_1} = \sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0.$$

All these quantities being positive they have a positive minimum c_6.

If we choose a positive number $c_4 < 1$ and $< c_6$, then the inequality $\lambda_h - \lambda_{h_1} > c_4$ will be true in either of the two considered cases. This proves the proposition of Part I.

Part II. If $\lambda_{h_2} \geqq 2\lambda_0 + 1$, then

$$\lambda_h^2 \geqq \sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + \lambda_{h_1}.$$

Proof: From $\lambda_{h_2} \geqq 2\lambda_0 + 1$ it follows $\lambda_{h_2} - 2\lambda_0 \geqq 1$ and

$$\lambda_{h_1} - \lambda_0 \geqq \lambda_{h_1} - 2\lambda_0 \geqq \lambda_{h_2} - 2\lambda_0 > 0.$$

We have

$$\lambda_h^2 = \left(\sum_{\tau=1}^{t} \lambda_{h_\tau} - \lambda_0\right)^2 \geqq \left(\sum_{\tau=1}^{t} \lambda_{h_\tau}\right)^2 - 2\lambda_0 \sum_{\tau=1}^{t} \lambda_{h_\tau} \geqq$$

$$\geqq \sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + 2\lambda_{h_1} \sum_{\tau=2}^{t} \lambda_{h_\tau} - 2\lambda_0 \sum_{\tau=1}^{t} \lambda_{h_\tau}.$$

Now

$$2\lambda_{h_1}\sum_{\tau=2}^{t}\lambda_{h_\tau}=\lambda_{h_2}\lambda_{h_1}+\lambda_{h_1}\lambda_{h_2}+2\lambda_{h_1}\sum_{\tau=3}^{t}\lambda_{h_\tau},$$

$$2\lambda_0\sum_{\tau=1}^{t}\lambda_{h_\tau}=2\lambda_0\lambda_{h_1}+2\lambda_0\lambda_{h_2}+2\lambda_0\sum_{\tau=3}^{t}\lambda_{h_\tau},$$

hence

$$\begin{aligned}2\lambda_{h_1}\sum_{\tau=2}^{t}\lambda_{h_\tau}-2\lambda_0\sum_{\tau=1}^{t}\lambda_{h_\tau}&=(\lambda_{h_2}-2\lambda_0)\lambda_{h_1}+(\lambda_{h_1}-2\lambda_0)\lambda_{h_2}\\&\quad+2(\lambda_{h_1}-\lambda_0)\sum_{\tau=3}^{t}\lambda_{h_\tau}\\&\geqq 1\,.\,\lambda_{h_1}+0+0.\end{aligned}$$

It follows

$$\lambda_h^2\geqq\sum_{\tau=1}^{t}\lambda_{h_\tau}^2+\lambda_{h_1}.$$

Part III. Proof of lemma 3.

In Part I we proved $\lambda_h-\lambda_{h1}>c_4$ for $h>r$; hence

$$\lambda_h^2-\lambda_{h_1}^2=(\lambda_h+\lambda_{h_1})(\lambda_h-\lambda_{h_1})>c_4\lambda_h.$$

It follows that $\lambda_h^2-\lambda_{h_1}^2$ tends to infinity when h increases indefinitely. We take a positive integer H, such that

$$H>r,$$

$$\left.\begin{aligned}\lambda_h^2-\lambda_{h_1}^2&>N(2\lambda_0+1)^2+|\bar{\lambda}_0|\\ \lambda_h&>N|\bar{\lambda}_0|\end{aligned}\right\}\text{ for } h\geqq H. \quad . \quad (18)$$

The last inequality implies

$$\lambda_{h_1}>|\bar{\lambda}_0| \qquad \text{for } h\geqq H, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

since $\lambda_{h1}\leqq|\bar{\lambda}_0|$ leads towards a contradiction because of

$$\lambda_h=\sum_{\tau=1}^{t}\lambda_{h_\tau}-\lambda_0\leqq t\,\lambda_{h_1}\leqq N|\bar{\lambda}_0|.$$

For $h<H$ there exist only a finite number of values for $|\bar{\lambda}_h|:\lambda_h^2$; we shall denote the largest of these quantities by c_7. If $c_8=c_7+1$, then

$$|\bar{\lambda}_h|<c_8\lambda_h^2\ ,\ c_8>1, \qquad \text{for } h<H.$$

In order to complete the proof of lemma 3 we shall show by induction that this inequality also holds for $h\geqq H$. Therefore we take $h\geqq H$ and suppose that the inequalities

$$|\bar{\lambda}_k|<c_8\lambda_k^2 \qquad \text{for } k=1,2,\ldots,h-1 \quad . \quad . \quad . \quad . \quad (20)$$

are true. Then we shall prove, that this relation also is satisfied for $k = h$.
By definition

$$\bar{\lambda}_h = \sum_{\tau=1}^{t} \bar{\lambda}_{h_\tau} - \bar{\lambda}_0 \ , \ h_\tau < h \qquad (h \geqq H > r),$$

hence in virtue of (20)

$$|\bar{\lambda}_h| < c_8 \sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + |\bar{\lambda}_0| ; \quad \ldots \ldots \ldots \quad (21)$$

here $|\bar{\lambda}_0| < \lambda_{h_1}$ by (19), therefore

$$|\bar{\lambda}_h| < c_8 \sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + \lambda_{h_1} < c_8 \sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + c_8 \lambda_{h_1} . \quad \ldots \ldots \quad (22)$$

At first we consider the special case

$$\lambda_{h_2} \geqq 2 \lambda_0 + 1,$$

when the condition of Part II is satisfied; it follows

$$\sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + \lambda_{h_1} \leqq \lambda_h^2 .$$

Therefore by (22)

$$|\bar{\lambda}_h| < c_8 \lambda_h^2 .$$

Hence in this case the inequality (20) also holds for $k = h$.
Now we consider the second possibility

$$\lambda_{h_2} < 2 \lambda_0 + 1 .$$

In this case we apply (21)

$$|\bar{\lambda}_h| < c_8 \sum_{\tau=1}^{t} \lambda_{h_\tau}^2 + |\bar{\lambda}_0| < c_8 \lambda_{h_1}^2 + c_8 (t-1) \lambda_{h_2}^2 + |\bar{\lambda}_0|$$

$$< c_8 \lambda_{h_1}^2 + c_8 N (2 \lambda_0 + 1)^2 + c_8 |\bar{\lambda}_0| .$$

Now $h \geq H$, hence $N(2\lambda_0 + 1)^2 + |\bar{\lambda}_0| < \lambda_h^2 - \lambda_{h_1}^2$ by (18); it follows

$$|\bar{\lambda}_h| < c_8 \lambda_{h_1}^2 + c_8 (\lambda_h^2 - \lambda_{h_1}^2) = c_8 \lambda_h^2 .$$

Therefore the inequality (20) also holds for $k = h$ in this second case.

This result proves the inequality $|\bar{\lambda}_h| < c_8 \lambda_h^2$ for any h and appropiately chosen c_8 and thus completes the proof of this lemma.

Lemma 4. If U denotes an arbitrary positive number, then there exists a positive number $\varepsilon = \varepsilon(U)$ with the following property:

Let the inequalities

$$|\bar{\lambda}_\varrho - \lambda_\varrho| < \varepsilon \qquad \text{for } \varrho = 0, 1, \ldots, r \quad \ldots \ldots \quad (6)$$

be satisfied, then it follows

$$|\bar{\lambda}_h| > U \qquad \text{f. s. l. } h.$$

Proof: Denoting the real part of a complex number z by $R(z)$, we shall even prove the stronger inequalities

$$R(\bar{\lambda}_h) > U \qquad \text{f. s. l. } h.$$

Without loss of generality we may suppose

$$U > \lambda_0 + N + 1. \quad \ldots \ldots \ldots \quad (23)$$

We take the positive integer $H = H(U)$ so large, that

$$H > r \text{ and } \lambda_H > (N+1)\,U. \quad \ldots \ldots \ldots \quad (24)$$

The remaining section of this proof we divide into two parts:

Part I: Let h_1 and h_2 be defined by the relations

$$\lambda_h = \sum_{\tau=1}^{t} \lambda_{h_\tau} - \lambda_0 \text{ , } h_\tau \geqq h_{\tau+1} \qquad (\tau = 1, 2, \ldots, t-1).$$

There exists a positive number $\varepsilon = \varepsilon(H)$, such that the inequalities

$$|\bar{\lambda}_\varrho - \lambda_\varrho| < \varepsilon \text{ for } \varrho = 0, 1, \ldots, r \text{ and } h_2 \leqq H$$

imply

$$R(\bar{\lambda}_h) > R(\bar{\lambda}_{h_1}) \text{ for } h > r.$$

Proof of Part I: For $h > r$ we have

$$\lambda_h - \lambda_{h_1} = \sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0 > 0, \qquad 2 \leqq t \leqq N.$$

If $h_2 \leqq H$, then $h_\tau \leqq H$ for $\tau = 2, 3, \ldots, t$, so that the number of different expressions $\sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0$ is finite; hence there exists a positive number $\delta = \delta(H)$, such that $\delta < 1$ and

$$h_2 \leqq H \text{ implies } \sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0 > \delta. \quad \ldots \ldots \ldots \quad (25)$$

Now

$$\lambda_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \lambda_\varrho \text{ , } \bar{\lambda}_h = \sum_{\varrho=0}^{r} g_{h\varrho}\, \bar{\lambda}_\varrho;$$

therefore

$$\bar{\lambda}_h - \lambda_h = \sum_{\varrho=0}^{r} g_{h\varrho} (\bar{\lambda}_\varrho - \lambda_\varrho).$$

Applying this result for $h = r+1, r+2, \dots, H$ we obtain: There exists a positive number $\varepsilon = \varepsilon(H)$, such that

$$|\bar{\lambda}_\varrho - \lambda_\varrho| < \varepsilon \text{ implies } |\bar{\lambda}_l - \lambda_l| < \frac{\delta}{N} \text{ for } l = 0, 1, \dots, H. \quad . \quad (26)$$

For $h > r$ we have

$$R(\bar{\lambda}_h - \bar{\lambda}_{h_1}) = R\left(\sum_{\tau=2}^{t} \bar{\lambda}_{h_\tau} - \bar{\lambda}_0\right) = \sum_{\tau=2}^{t} R(\bar{\lambda}_{h_\tau}) - R(\bar{\lambda}_0) =$$

$$= \sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0 + \sum_{\tau=2}^{t} R(\bar{\lambda}_{h_\tau} - \lambda_{h_\tau}) - R(\bar{\lambda}_0 - \lambda_0),$$

$$R(\bar{\lambda}_h - \bar{\lambda}_{h_1}) \geqq \sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0 - \sum_{\tau=2}^{t} |\bar{\lambda}_{h_\tau} - \lambda_{h_\tau}| - |\bar{\lambda}_0 - \lambda_0|.$$

If the conditions of the assertion in Part I are satisfied, then $h_2 \leqq H$, hence $\sum_{\tau=2}^{t} \lambda_{h_\tau} - \lambda_0 > \delta$ by (25); further $|\bar{\lambda}_\varrho - \lambda_\varrho| < \varepsilon$, $h_\tau \leqq H$ for $\tau \geqq 2$, hence $|\bar{\lambda}_{h_\tau} - \lambda_{h_\tau}| < \frac{\delta}{N}$ by (26) for $\tau \geqq 2$; also $|\bar{\lambda}_0 - \lambda_0| < \frac{\delta}{N}$.

It follows for $h > r$

$$R(\bar{\lambda}_h - \bar{\lambda}_{h_1}) > \delta - (t-1)\frac{\delta}{N} - \frac{\delta}{N} \geqq 0.$$

This gives the desired result.

Part II: Proof of $R(\bar{\lambda}_h) > U$ for $h \geqq H$.

We observe, that

$$R(\bar{\lambda}_H) = \lambda_H + R(\bar{\lambda}_H - \lambda_H) \geqq \lambda_H - |\bar{\lambda}_H - \lambda_H|;$$

further that $\lambda_H > (N+1)\,U$ by (24) and $|\bar{\lambda}_H - \lambda_H| < \frac{\delta}{N} < 1$ by (26).

It follows

$$R(\bar{\lambda}_H) > (N+1)\,U - 1 > U.$$

Hence the assertion of Part II is true for $h = H$.

Now we take $h > H$ and we suppose

$$R(\bar{\lambda}_l) > U \text{ for } l = H, H+1, \dots, h-1. \quad . \quad . \quad . \quad . \quad (27)$$

Then we shall show that this inequality even holds for $l = h$.

Taking account of $h > H > r$ we have

$$\lambda_h = \sum_{\tau=1}^{t} \lambda_{h_\tau} - \lambda_0,$$

where $h_1 \geqq h_2 \geqq \dots \geqq h_t$. Let h_s be the first number in this sequence $\leqq H$ (if there exists not a number with this property, then we put $s = t+1$):

$$h > h_1 \geqq h_2 \geqq \dots \geqq h_{s-1} > H \geqq h_s \geqq \dots \geqq h_t.$$

We have

$$\bar{\lambda}_h = \sum_{\sigma=1}^{s-1} \bar{\lambda}_{h_\sigma} + \sum_{\tau=s}^{t} \bar{\lambda}_{h_\tau} - \bar{\lambda}_0,$$

$$R(\bar{\lambda}_h) = \sum_{\sigma=1}^{s-1} R(\bar{\lambda}_{h_\sigma}) + \sum_{\tau=s}^{t} R(\bar{\lambda}_{h_\tau}) - R(\bar{\lambda}_0)$$

$$= \sum_{\sigma=1}^{s-1} R(\bar{\lambda}_{h_\sigma}) + \sum_{\tau=s}^{t} \lambda_{h_\tau} - \lambda_0 + \sum_{\tau=s}^{t} R(\bar{\lambda}_{h_\tau} - \lambda_{h_\tau}) - R(\bar{\lambda}_0 - \lambda_0),$$

$$R(\bar{\lambda}_h) \geqq \sum_{\sigma=1}^{s-1} R(\bar{\lambda}_{h_\sigma}) + \sum_{\tau=s}^{t} \lambda_{h_\tau} - \lambda_0 - \sum_{\tau=s}^{t} |\bar{\lambda}_{h_\tau} - \lambda_{h_\tau}| - |\bar{\lambda}_0 - \lambda_0|.$$

It follows from (27), that $R(\bar{\lambda}_{h_\sigma}) > U$ is valid, since $h > h_\sigma > H$; now $|\bar{\lambda}_\varrho - \lambda_\varrho| < \varepsilon$ is given; further $h_\tau \leqq H$ ($\tau = s, \ldots, t$) by definition, hence $|\bar{\lambda}_{h_\tau} - \lambda_{h_\tau}| < \dfrac{\delta}{N} < 1$ by (26); also $|\bar{\lambda}_0 - \lambda_0| < \dfrac{\delta}{N} < 1$.

Therefore we obtain

$$R(\bar{\lambda}_h) > (s-1)\,U + \sum_{\tau=s}^{t} \lambda_{h_\tau} - \lambda_0 - N - 1. \quad \ldots \quad (28)$$

We consider the following three possibilities separately:
1) $s = 1$, 2) $s = 2$ and 3) $s \geqq 3$.
1) $s = 1$ ($h_1 \leqq H$). Then by (28)

$$R(\bar{\lambda}_h) > \sum_{\tau=1}^{t} \lambda_{h_\tau} - \lambda_0 - N - 1 = \lambda_h - N - 1.$$

Now $h > H$, hence $\lambda_h > (N+1)\,U$ by (24); therefore

$$R(\bar{\lambda}_h) > (N+1)\,U - N - 1 > U,$$

since $U > N + 1$ by (23).

2) $s = 2 : h_1 > H$; $h_2 \leqq H$. We showed in Part I for this special case $R(\bar{\lambda}_h) > R(\bar{\lambda}_{h_1})$ (since $h > H > r$).

It follows from (27) $R(\bar{\lambda}_{h_1}) > U$ since $h > h_1 > H$. Hence

$$R(\bar{\lambda}_h) > R(\bar{\lambda}_{h_1}) > U.$$

3) $s \geqq 3$. In this case the inequality (28) gives

$$R(\bar{\lambda}_h) \geqq 2\,U - \lambda_0 - N - 1 > U,$$

since $U > \lambda_0 + N + 1$ by (23).

In all three cases the inequality $R(\bar{\lambda}_h) > U$ is satisfied.

This result proves the lemma.

Lemma 5: Let

$$P_k(x_1, x_2, \ldots, x_n) = 0 \qquad (k = 1, 2, \ldots)$$

denote a system of algebraic equations with rational coefficients and with one solution (X_ν) at least. If ε is an arbitrary positive number, then there exists also a solution $(\overline{X}_\nu)$ in algebraic numbers, such that

$$|\overline{X}_\nu - X_\nu| < \varepsilon \qquad (\nu = 1, 2, \ldots, n).$$

Proof: If all numbers X_ν are algebraic, then there is nothing to prove. Hence we may assume that at least one of these numbers is transcendental. Let r denote in this lemma the highest number of algebraic independent quantities in the set $X_1, X_2, \ldots, X_n$ $(1 \leq r \leq n)$. Without loss of generality we may suppose, that $X_1, X_2, \ldots, X_r$ are independent: no polynomial in $X_1, X_2, \ldots, X_r$ with rational coefficients is zero unless these coefficients themselves are all zero. Therefore $X_1, X_2, \ldots, X_r$ are independent indeterminates over the field $\boldsymbol{P}$ of all rational numbers, whereas the remaining numbers $X_{r+1}, X_{r+2}, \ldots, X_n$ are algebraic over the field $\boldsymbol{P}(X_1, X_2, \ldots, X_r)$. We shall denote these last $n - r$ quantities by $Y_1, Y_2, \ldots, Y_m$ $(m = n - r)$.

For Y_1 there exists an irreducible equation over $\boldsymbol{P}(X_\varrho)$:

$$Y_1^{g_1} + A_{11}(X_\varrho)\, Y_1^{g_1 - 1} + \ldots + A_{1 g_1}(X_\varrho) = 0,$$

where $A_{1\gamma}(X_\varrho)$ are rational functions in $X_1, X_2, \ldots, X_r$ with rational coefficients $(\gamma = 1, 2, \ldots, g_1)$.

Now we adjoin Y_1 to the field $\boldsymbol{P}(X_\varrho)$; Y_2 certainly is algebraic over $\boldsymbol{P}(X_\varrho; Y_1)$, hence there exists an irreducible equation

$$Y_2^{g_2} + A_{21}(X_\varrho; Y_1)\, Y_2^{g_2 - 1} + \ldots + A_{2 g_2}(X_\varrho; Y_1) = 0,$$

where $A_{2\gamma}(X_\varrho; Y_1)$ are rational functions in $X_1, X_2, \ldots, X_r, Y_1$ with rational coefficients $(\gamma = 1, 2, \ldots, g_2)$. Y_1 being algebraic over $\boldsymbol{P}(X_\varrho)$, we may assume that the denominators in all these g_2 rational functions $A_{2\gamma}$ are polynomials in $X_1, X_2, \ldots, X_r$ only.

In such a manner we may proceed: For Y_μ $(\mu = 1, 2, \ldots, m)$ there exists an irreducible equation in $\boldsymbol{P}(X_\varrho; Y_1, \ldots, Y_{\mu-1})$

$$Y_\mu^{g_\mu} + A_{\mu 1}(X_\varrho; Y_1, \ldots, Y_{\mu-1})\, Y_\mu^{g_\mu - 1} + \ldots + A_{\mu g_\mu}(X_\varrho; Y_1, \ldots, Y_{\mu-1}) = 0, \quad (29)$$

where $A_{\mu\gamma}(X_\varrho; Y_1, \ldots, Y_{\mu-1})$ are rational functions in $X_\varrho; Y_1, Y_2, \ldots, Y_{\mu-1}$ with rational coefficients. We may assume that the denominators in these functions are polynomials in $X_1, X_2, \ldots, X_r$ only. We denote the least common multiple of all occurring denominators by

$$V(X_1, X_2, \ldots, X_r).$$

being a polynomial with rational coefficients in $X_1, X_2, \ldots, X_r$ only.

The set $(X_\varrho; Y_\mu)$ is a solution of the system of equations

$$\left.\begin{aligned}
&y_1^{g_1} + A_{11}(x_\varrho)\, y_1^{g_1 - 1} + \ldots + A_{1 g_1}(x_\varrho) = 0,\\
&y_2^{g_2} + A_{21}(x_\varrho; y_1)\, y_2^{g_2 - 1} + \ldots + A_{2 g_2}(x_\varrho; y_1) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&y_m^{g_m} + A_{m 1}(x_\varrho; y_1, \ldots, y_{m-1})\, y_m^{g_m - 1} + \ldots + A_{m g_m}(x_\varrho; y_1, \ldots, y_{m-1}) = 0,
\end{aligned}\right\} \quad (30)$$

we obtain from (29) by replacing the quantities $X_\varrho; Y_\mu$ by unknowns $x_\varrho; y_\mu$.

Now we apply a theorem of E. NOETHER [8]):

„Let $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_r$ be arbitrary algebraic numbers, such that $V(\overline{X}_\varrho) \neq 0$ and let $\overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_m$ denote (algebraic) numbers, such that $(\overline{X}_\varrho; \overline{Y}_\mu)$ is a solution of (30).

If $P(x_1, x_2, \ldots, x_r; y_1, y_2, \ldots, y_m)$ is a polynomial with rational coefficients, then $P(X_\varrho; Y_\mu) = 0$ implies $P(\overline{X}_\varrho, \overline{Y}_\mu) = 0$."

Applying this theorem on every polynomial $P_k(x_\nu)$ of the sequence in this lemma $(k = 1, 2, \ldots)$, we obtain the following result:

If $(\overline{X}_\varrho; \overline{Y}_\mu)$ is a solution of the system (30) in algebraic numbers, such that $V(\overline{X}_\varrho) \neq 0$, then it also is a solution for the system $P_k(x_\nu) = 0$ $(k = 1, 2, \ldots)$.

The only remaining problem is to prove the existence of algebraic numbers $\overline{X}_\varrho; \overline{Y}_\mu$, satisfying (30) and with

$$V(\overline{X}_\varrho) \neq 0, \; |\overline{X}_\varrho - X_\varrho| < \varepsilon, \; |\overline{Y}_\mu - Y_\mu| < \varepsilon.$$

We consider a sequence of points $(x_{h1}, x_{h2}, \ldots, x_{hr})$ in the r-dimensional space $(h = 1, 2, \ldots)$, such that all coordinates are algebraic numbers and such that

$$\lim_{h \to \infty} (x_{h\varrho}) = (X_\varrho),$$

so that for an arbitrary positive number δ the inequalities $|x_{h\varrho} - X_\varrho| < \delta$ are satisfied f.s.l. h. It is clear that $V(x_{h\varrho}) \neq 0$ f.s.l. h, since $V(X_\varrho) \neq 0$.

Now we apply the classical theorem on the continuity of the roots of an algebraic equation. Y_1 is a root of the equation

$$y_1^{g_1} + A_{11}(x_\varrho)\, y_1^{g_1-1} + \ldots + A_{1g_1}(x_\varrho) = 0, \quad \ldots \quad (31)$$

when X_ϱ is substituted for x_ϱ. Let δ_1 be an arbitrary positive number, then there exist a positive number δ_2 with the following property: For any given point (x_ϱ) with $|x_\varrho - X_\varrho| < \delta_2$ we can choose a number y_1, such that (x_ϱ, y_1) is a solution of the equation (31) and $|y_1 - Y_1| < \delta_1$. If all coordinates x_ϱ are algebraic then also y_1 is algebraic.

Hence we obtain in the $(r+1)$-dimensional space a sequence of "algebraic" points $(x_{h\varrho}; y_{h1})$, such that the coordinates of each point satisfy the equation (31) when $x_{h\varrho}$ is substituted for x_ϱ and y_{h1} for y_1, and such that

$$\lim_{h \to \infty} (x_{h\varrho}; y_{h1}) = (X_\varrho; Y_1).$$

Repeating this argument we finally prove the existence of a sequence of algebraic points $(x_{h\varrho}; y_{h1}, \ldots, y_{hm})$, such that the coordinates of each point satisfy the equations of the system (30) when $x_{h\varrho}$ is substituted for x_ϱ and $y_{h\mu}$ for y_μ and such that

$$\lim_{h \to \infty} (x_{h\varrho}; y_{h1}, \ldots, y_{hm}) = (X_\varrho; Y_1, \ldots, Y_m).$$

This result proves the lemma.

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8) See: B. L. v. d WAERDEN, Moderne Algebra II, second ed, p. 50—52.

Mathematics. — *An analogue of the nine-point circle in the space of n-dimensions.* By J. C. H. GERRETSEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 27, 1945.)

1. Let the vertices of a simplex in the space of n dimensions be A_0, A_1, A_2, ..., A_n. It can be assumed that each edge $A_i A_k$ is perpendicular to the opposite $(n-2)$-dimensional boundary face, so the altitude lines through the vertices are concurrent in a point called the orthocentre of the simplex. Moreover the same peculiarity holds for every $(n-1)$-face of the simplex and the orthogonal projection of the orthocentre on any $(n-1)$-face is just the orthocentre of that face. In the trivial case $n = 2$ every point of the line $A_i A_k$ must be regarded as an orthocentre of the 1-simplex with edges A_i and A_k.

In the case of an orthocentric simplex a generalisation of the nine-point circle of a triangle has been given by R. MEHMKE, Arch. f. Math. u. Phys. **70** (1884), p. 210. It is our aim to give an analogue in the general case. In that case the altitude lines have not a point in common, but — as we shall see — there will be a point that has analogous properties as the orthocentre in the special case mentioned above.

2. Let M_{ik} be the middle point of the edge $A_i A_k$ and G_{ik} the centre of gravity of the opposite $(n-2)$-face A_{ik}^{n-2}. If G be the centre of gravity of the n-simplex, it is easily proved that the points M_{ik}, G and G_{ik} are collinear, the segment $M_{ik} G_{ik}$ being divided by G in the ratio $(n-1) : 2$.

The $n+1$ hyperplanes each going through the points M_{ik} and perpendicular to the lines $A_i A_k$ are concurrent in a point M, the centre of the circumscribed hypersphere of the n-simplex. By a similitude with centre G and with ratio $-2 : (n-1)$ these hyperplanes are transformed into the hyperplanes through the points G_{ik}, each being perpendicular to the opposite edge $A_i A_k$. *Hence these hyperplanes are concurrent in a point H.* In the case $n = 3$ this is the well-known point of MONGE; we will denote it as the point of MONGE in the general case also. The result can be formulated in the following way: *the centre of gravity G is collinear with the centre of the circumscribed hypersphere M and the point of* MONGE *H and divides the segments $M H$ in the ratio* $(n-1) : 2$. It is easily seen that in the case of the orthocentric simplex the point of MONGE and the orthocentre are coincident.

3. Let H_k be the point of MONGE of the $(n-1)$-face A_k^{n-1} opposite to the vertex A_k, $(k = 0, 1, ..., n)$. Let $'A_k$ be the orthogonal projection of

A_k and $'H_k$ the orthogonal projection of H on that face. *We will prove that the points $'A_k$, $'H_k$ and H_k are collinear and that $'H_k$ divides the segment $'A_k H_k$ in the ratio* $(n-2):1$. It is to be noted that the theorem is trivial if $n \leqq 2$.

To prove the theorem we regard at first the edge $A_0 A_1$. The point H_n is laying in the $(n-2)$-space perpendicular to $A_0 A_1$ which is passing through the centre of gravity of the $(n-3)$-dimensional boundary simplex of the simplex $A_0 \dots A_{n-1}$ opposite to $A_0 A_1$ and contained in that simplex. A similitude with centre $'A_n$ and ratio $(n-2):(n-1)$ transforms this space into an $(n-2)$-dimensional space perpendicular to $A_0 A_1$ going through the centre of gravity $'G_{01}$ of the simplex $A_2 \dots A_{n-1}\, 'A_n$. But this point is just the orthogonal projection of G_{01} on the space through the points $A_0 \dots A_{n-1}$ and therefore the $(n-2)$-space just mentioned is the intersection of the $(n-1)$-space through $A_0, \dots, A_{n-1}$ and the hyperplane through G_{01} perpendicular to $A_0 A_1$. This space cuts the line $'A_n H_n$ in the point $'H_n$ and the segment $'A_n H_n$ is divided by $'H_n$ in the ratio $(n-2):1$. If we take instead of $A_0 A_1$ any other edge of the $(n-1)$-simplex $A_0 \dots A_{n-1}$, we always find the same point $'H_n$. Therefore the point of MONGE from the n-simplex must be situated on the line through $'H_n$ perpendicular to the hyperplane through $A_0, A_1, \dots, A_{n-1}$, q.e.d.

4. Now we are able to give an analogue of the nine-point circle in the following manner: Let G_k denote the centre of gravity of the $(n-1)$-dimensional face opposite to the vertex A_k, P_k the point on the segment $A_k H$ on a distance $\frac{1}{n} A_k H$ from H, $'P_k$ the harmonic conjugate of H_k with regard to $'A_k$ and $'H_k$. *Then the* $2(n+1)$ *points* G_k, P_k, $'P_k$ *are on one hypersphere. The centre N of this hypersphere is the harmonic conjugate of M with regard to G and H.*

In proving this theorem we see that a similitude with centre G and ratio $-1:n$ transforms the circumscribed hypersphere of the n-simplex into the same hypersphere as is obtained by a similitude with centre H and ratio $1:n$. The points P_k and G_k are diametrically situated on this sphere and — if $''P_k$ denotes the orthogonal projection of P_k on the face opposite to A_k — $\angle P_k\, ''P_k\, G_k$ is a right angle; hence $''P_k$ is also on that sphere. An easy computation shows that $''P_k$ is the harmonic conjugate of H_k with regard to $'A_k$ and $'H_k$, so the points $''P_k$ and $'P_k$ coalesce.

Physiology. — *Over den invloed van groeihormoon op den skeletspiergroei en op de stikstofuitscheiding.* By J. H. GAARENSTROOM and A. KRET. (Uit het Pharmacologisch Laboratorium der Rijksuniversiteit te Leiden, Beheerend directeur: Dr. S. E. DE JONGH.) (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of November 24, 1945.)

Reeds eenige malen berichtten wij over onze onderzoekingen betreffende het aangrijpingspunt van groeihormoon. De mogelijkheid dat groeihormoon den groei van organen of orgaangroepen afzonderlijk, dus door meer dan één aangrijpingspunt (al dan niet gecoördineerd), zou regelen, werd door ons systematisch nagegaan. Daarbij kwam aan het licht, dat voor het aannemen van een directen invloed op den groei van de *ingewanden* geen enkele reden aanwezig was (GAARENSTROOM a) en dat zelfs een bijzondere werking op den *skelet*groei wel is waar niet uit te sluiten was, maar toch wel als onbewezen moest worden beschouwd (BOERÉ en GAARENSTROOM, GAARENSTROOM b).

Wij onderzochten verder den invloed van groeihormoon op den groei van de skelet*spieren*, en wel de daarbij door ons vroeger ook voor skelet en ingewanden gestelde, min of meer principieele, vraag: is groei van skeletspieren mogelijk bij afwezigheid van groeihormoon?

De methode bij deze proeven was als volgt. Een aantal ratten van 100—130 gram lichaamsgewicht werd gehypophysectomeerd. Een week (deze periode werd ingeschakeld om het spierstelsel gelegenheid te geven tot de na hypophysectomie gebruikelijke atrophie) na deze operatie werd aan den linkerpoot de m. gastrocnemius aan zijn insertie doorgesneden, waarvan het gevolg was, dat aan de Achillespees nu nog slechts de kleine m. soleus insereerde. Laatstgenoemde spier moest daardoor de functie van eerstgenoemde overnemen, wat een prikkel moet zijn tot hypertrophie. Tien dagen na de doorsnijding werd sectie verricht en het eventueele ontstaan van zulk een hypertrophie beoordeeld, door het gewicht van den m. soleus aan de geopereerde zijde te vergelijken met dat aan de andere zijde.

Deze proef, die met verschillende series ratten werd verricht, en waarbij niet gehypophysectomeerde ratten, bij welke dezelfde doorsnijding werd gedaan, als contrôle dienden, leverde de in tabel I neergelegde resultaten op. In alle gevallen was de m. soleus aan de doorgesneden zijde zwaarder dan die aan de niet geopereerde. Het gemiddelde gewichtsverschil bedroeg 24 %. Dat beteekent dus een hypertrophie, althans in relatieven zin. Deze kan bij de hypophyselooze dieren niet geheel als werkelijke spier*groei* worden beschouwd, daar de m. soleus aan de contrôlezijde gedurende de betreffende periode waarschijnlijk in gewicht achteruitging.

Daar, zooals vroeger werd vastgesteld, deze daling parallel met die van het lichaamsgewicht verloopt, zou zij ongeveer 8 % hebben bedragen. Het grootste deel van het gewichtsverschil tusschen den linker en rechter m. soleus moet dan dus toch op *groei* van de spier aan de geopereerde zijde hebben berust. Dit beteekent, dat dezelfde aanleiding, die bij normale dieren groei van de spieren veroorzaakt, dit ook nog doet bij afwezigheid van groeihormoon. Het gewichtsverschil was zelfs niet minder bij de niet gehypophysectomeerde dieren, hoewel hierbij overwogen moet worden, dat bij de laatste de contrôlespieren tijdens de proef groeiden, terwijl deze bij de hypophyselooze dieren achteruitging.

Het groeihormoon blijkt dus geen onontbeerlijke factor voor groei van skeletspieren te zijn, wat ons inziens, vooral omdat andere, op den groei van speciale organen aangrijpende (organotrope) hypophysehormonen dit, voor het betreffende orgaan, wèl zijn, tegen zulk een aangrijpingspunt voor het groeihormoon pleit. Hetzelfde en andere argumenten maakten vroeger reeds een directen invloed op den groei van het skelet en de inwendige organen onwaarschijnlijk en de uitkomsten van het bovenbeschreven onderzoek sterkt ons in de meening, dat het groeihormoon een algemeenen, niet tot afzonderlijke organen gerichten, invloed heeft, die tot lichaamsgroei leidt. Voortbouwende hierop, komt men er, zooals ook vroeger reeds beschreven (BOERÉ en GAARENSTROOM) toe, voor het groeihormoon een plaats te zoeken onder de niet-organotrope hypophysaire hormonen, dus bij de groep, die aangrijpt in de stofwisseling.

TABEL I.

Aantal dieren	Hypl. of norm.	Gem. lich. gew. (gr.)			Gem. gewicht m. soleus (mgr.)				% verschil	
		Bij hypoph.	Bij spierdoorsnijding	Bij sectie	Geoper. zijde		Niet geop. zijde			
					Gem.	Uitersten	Gem.	Uitersten		
5	hpl	128	115	103	53	46—64	40	35—47	33	
8	„	129	118	107	57	49—64	47	42—54	18	24
7	„	144	126	119	62	49—82	46	37—57	26	
10	„	149	137	128	65	50—81	52	43—66	20	
10	norm.		115	123	56	39—82	42	35—54	24	

Zonder op dit punt stelling te nemen ten gunste van de door sommige Amerikaansche onderzoekers (LEE en SHAFFER, GAEBLER) geuite hypothese, dat het groeihormoon een eiwitstofwisselingshormoon zou zijn, kan hierbij toch worden opgemerkt, dat deze meening de tot nu toe bekende feiten op aangename wijze zouden kunnen verklaren. LEE en SHAFFER vinden, dat de door hen en anderen (GAEBLER, HARRISON en LONG) gevonden vermindering van de stikstofuitscheiding en het gehalte aan aminozuren in het bloed vóór hun opvatting pleiten; deze verschijnselen kunnen echter even goed het indirecte gevolg van een dieper schuilenden invloed van het groeihormoon zijn. Toch lijkt deze vondst als uitgangspunt

voor verder onderzoek van belang. Wij meenden daarom, dat het nuttig was, enkele der betreffende proeven te herhalen en uit te breiden. Wat dit laatste betreft, ten eerste waren de vroegere experimenten verricht met vrij weinig gezuiverde groeihormoonpraeparaten, terwijl wij met een sterk gezuiverd praeparaat [1]) practisch vrij van andere bekende hypophysehormonen werkten. Ten tweede hebben wij ook proeven gedaan met dieren (ratten van ca. 100 gram), die een *eiwitloos dieet* (bestaande uit 64 % koolhydraten, 6 % vet, 27 % water, wat vaseline e.d.) ontvingen. Het daarnaast gebruikte eiwithoudende dieet had dezelfde samenstelling, behoudens dat 25 % in plaats van uit koolhydraten uit eiwit bestond. Drie groepen van te voren gehypophysectomeerde dieren werden vergeleken; één groep ontving eiwithoudend dieet, een tweede groep eiwitloos dieet, een derde eiwitloos dieet en tevens 20 E groeihormoon per dag. De voedseltoediening was beperkt tot 6 gram van het juist beschreven voedsel per dag, welke hoeveelheid door alle dieren practisch geheel werd gegeten. Nagegaan werd de stikstofuitscheiding in de urine van elke rat afzonderlijk in 3 perioden, n.l. de eerste 48 uur na hypophysectomie, de daarop volgende 72 uur en de daar weer op volgende 48 uur. Het stikstofgehalte werd bepaald met de methode van KJELDAHL.

De resultaten beantwoordden, gelijk tabel II laat zien, in belangrijke

TABEL II.

Aantal dieren	Lich. gewicht (gr.)			Voedsel en behandeling		N uitsch. in urine (gem.) (mgr.)				Orgaangew. (gem.) (mgr.)		
	Bij hypoph.	Na een week	% toeneming	Gegeven (per week)	Gegeten p. w.	48 uur	72 uur	48 uur	Totaal	Lever	Nieren (2)	M. gastrocnemii
7	140	108	−23	42 gr. E.H. voer	37	145	197	109	451	3870	780	1260
7	139	111	−20	42 gr. E.L voer	35	116	157	85	358	3800	814	1266
6	134	112	−16	42 gr. E.L. voer + Gr. [3])	38	97	99	50	246	3960	840	1260
9	111	90	−19	42 gr. E.H. voer	40					3550	711	984 [2])
7	112	90	−20	56 [4]) gr. E.L. voer	47					3520	693	1010 [2])
9	113	93	−18	56 [4]) gr. E.L. voer + Gr [3])	39					3450	713	1072 [2])
6	120	94	−22	42 gr. E.H. voer	41					3660	732	1132
8	120	98	−18	42 gr. E.L. voer	38					3870	674	1130
6	118	94	−20	42 gr. E.L. voer + Gr. [3])	37					3480	770	1130
3	128	110	−14	42 gr. E.H. voer	38					4190	924	1306
4	127	105	−17	42 gr. E.L. voer	38					3610	838	1280

opzichten aan de op grond van de literatuur gestelde verwachtingen. De stikstofuitscheiding is minder bij de met groeihormoon behandelde

[1]) 1 $E = 50\,\gamma$. Welwillend afgestaan door Dr. E. DINGEMANSE.

[2]) Eiwitgehalte spier resp.: 192, 196 en 204 mg.

[3]) Groeihormoon, dosis altijd 20 E per dag.

[4]) Deze 56 gram waren (dit was de eerste proef) oriënteerend; het bleek, dat de dieren van het eiwitlooze voer niet meer dan gemiddeld 42 gram aten, daarom werd later niet meer dan 42 gr gegeven.

ratten, dan bij de eveneens eiwitloos gevoede contrôles. Bij de eiwithoudend gevoede, niet ingespoten dieren was de uitscheiding nog grooter, wat in verband met de eiwitopneming geen verwondering behoeft te wekken. Verscheidene andere conclusies, b.v. ten aanzien van den invloed van groeihormoon op den opbouw en de afbraak van eiwit, zouden aan de uitkomsten kunnen worden vastgeknoopt, ware het niet, dat zich hierbij een moeilijkheid vóórdeed: De hoeveelheid uitgescheiden stikstof was n.l. belangrijk minder dan men op grond van het eiwitverlies (geschat met behulp van lichaams- en orgaangewichten) zou verwachten. Een bij benadering volledige stikstofbalans geleek ons voor vèrgaande conclusies noodzakelijk. Pogingen om tot dit doel te geraken met behulp van nieuwe proeven, moesten door de oorlogsomstandigheden worden gestaakt. Dit laatste is ook de reden waarom het onderzoek in dit vroege stadium reeds wordt gepubliceerd.

De stikstofretentie, welke wel niet anders dan een eiwitretentie beteekent, veroorzaakt door groeihormoon, zou zich tenslotte moeten uiten in een grooter orgaangewicht. De orgaangewichten van de drie groepen en van nog andere series op dezelfde wijze behandelde dieren verschilden, zooals tabel II eveneens laat zien, niet belangrijk. Bij één der proeven werden ook de eiwitgehalten der m. gastrocnemii bepaald volgens een vroeger beschreven methode (KRET en DE JONGH); ook wat dit betreft werden geen verschillen tusschen de groepen gevonden. Vermoedelijk was de duur van de proef te kort en de maatstaf te weinig gevoelig om dergelijke verschillen aan den dag te doen treden.

Samenvatting.

Doorsnijding van de insertie van den m. gastrocnemius heeft tot gevolg, dat de kleine, daaronder gelegen m. soleus de functie van beide spieren overneemt en hypertrophieert. Deze operatie werd door ons aan één zijde verricht bij te voren gehypophysectomeerde ratten. Het bleek, dat ook onder deze omstandigheden de hypertrophie, dus de groei van spierweefsel, nog plaats vond. Het hypophysaire groeihormoon is daarvoor dus niet noodzakelijk.

De toediening van groeihormoon aan gehypophysectomeerde, eiwitloos gevoede dieren, doet de uitscheiding van stikstof in de urine, ten opzichte van niet ingespoten contrôledieren, dalen.

Zusammenfassung.

1) Durchtrennung der Insertion des M. gastrocnemius ruft hervor, dass der kleine, darunten gelegene, Soleusmuskel die Gesamt-Funktion übernimmt und hypertrophiert. Wir übten diese Operation einseitig bei vorher hypophysektomierten Ratten. Es ergab sich, dass auch unter diesen Verhältnissen die Hypertrophie, also ein Wachstum von Skeletmuskelgewebe, stattfindet. Das Wachstumshormon der Hypophyse ist dabei nicht erforderlich.

2) Zufuhr von Wachstumshormon an hypophysenlose, eiweissfrei ernährte Ratten, gab eine Senkung des Harn-Stickstoffs, bezogen auf nicht gespritzte, ebenfalls eiweisslos ernährte, hypophysenlose Kontroltiere.

Summary.

After cutting the m. gastrocnemius at the place of insertion, the underlying m. soleus takes over the task of the former muscle and becomes hypertrophic. When this operation is performed upon hypophysectomized rats, the hypertrophy still arises. So the pituitary growth hormone is not necessary for the growth of muscle tissue.

When growth hormone is administered to hypophysectomized rats, fed on a diet free from protein, the output of nitrogen in the urine decreases.

Résumé.

Après le détachement de l'insertion du muscle gastrocnémien, le muscle solaire, se trouvant en dessous, prend la fonction des deux muscles en hypertrophiant. Nous avons fait cette opération unilatéralement chez des rats hypophysectomisés d'avance. Sous ces circonstances l'hypertrophie (donc la croissance du tissu musculaire) se manifesta encore; alors elle n'a pas besoin de l'hormone de croissance hypophysaire.

L'administration de l'hormone de croissance à des rats hypophysectomisés étant nourris sans protéine, fait diminuer l'excrétion d'azote dans l'urine à l'égard des rats de contrôle non injectés.

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